SOME RESULTS ON TRIANGLE CONTRACTIVE MAPS AND ORTHOGONALITY IN NORMED LINEAR SPACES

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CERTIFICATE

This is to certify that the research work embodied in the thesis entitled 'Some Results on Triangle Contractive Maps and Orthogonality in Normed Linear Spaces' by S.B. Mathur, a Ph.D. student of this department, has been carried out under my supervision and that it has not been submitted elsewhere for any degree.

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CONTENTS

CHAPTER 0

BRIEF REVIEW, PRELIMINARIES AND BASIC RESULTS

0.1. We begin by giving the background for our thesis and outlining the main results contained therein. The definitions and notations and the results referred to are given in Section 2 of this chapter.

Daykin [7] raised the problem of characterizing self maps on a Hilbert space preserving linearity (Collineations), that is the map under which the images of collinear points are collinear. Daykin [8] himself proved that on a real Euclidean plane E every non-trivial (i.e. T(E) not a subset of a line) continuous collineation map T is affine, and the result was used to characterize the maps $T: E \to E$ for which there is an $\alpha > 0$ such that $\Delta(Tx, Ty, Tz) \leq \alpha \Delta(x, y, z)$ for all $x, y, z \in E$. Here $\Delta(x, y, z)$ denotes the area of a triangle formed by x, y and z. We may point out here that a similar mapping on subsets of R^n was considered by Zamfirescue [32] also.

In [10], Daykin and Dugdale introduced a more general concept of triangle contractive (TC) self maps of a Hilbert space (def. 0.2.1(ii)). They have also given, therein, various sufficient conditions for a TC map to have a fixture (Theorems 0.2.2(i)-(v)). In the same paper they conjectured that 'Every TC self map has a fixture (when the space is finite-dimensional)'.

For a plane the conjecture was proved by Daykin [9], and in [1] Ang and Hoa proved the same for an n-dimensional space. In the first chapter we have shown that even in an infinite-dimensional Hilbert space H a wide class of TC maps have fixtures. This class includes for example non-expansive TC maps, compact TC maps, TC maps which are completely continuous perturbations of non-expansive maps and all such maps as are pseudo-contractive. In fact, we have proved that all continuous bounded TC maps are 1-ball contractive which has led us to conjecture that all TC maps have fixtures. Our main result enables us to extend to infinite-dimensional spaces some of the results of [10] concerning fixtures under the assumptions on f such as 'there exists a sequence $\{x_n\}$ such that $\|x_n - f\, x_n\| \to 0$' or 'there exists a sequence of iterates of f converging to a line'

Rhoades [27] has given a definition of a generalized triangle contractive (GTC) map f on a Hilbert space H (Definition 0.2.1 (iii)), such that every TC map is a GTC map. Proceeding with this definition Rhoades could prove some of the results of [10] such as 'if f has no fixed point then it has atmost one fixed line, and if it has one such line L then $\{f^n w\}$ converges to L for every $w \in H$' or 'if there is a point p such that every neighbourhood of p contains a point x and its image fx, then either p is a fixed point or L(p, fp) is a fixed line containing f H ' or 'if H is finite-dimensional and f has a sequence of iterates which converges to a line then f has a fixture'. By an example he has shown that a discontinuous GTC map may not

have a fixture, whereas it is known that a discontinuous TC map necessarily has a fixture. Continuity plays an important role in our discussion of the existence of fixtures in general, which is our main theme in a greater part of this thesis. Therefore, we have confined ourselves to TC maps.

The second chapter dwells on the tetrahedron contractive (TTC) self maps of a Hilbert space (Definition 0.2.3(ii)) - about which Daykin and Dugdale [10] expressed the belief that 'if tetrahedrons go down then the self map will have a fixed point, line or plane and so on'. However, it seems that this investigation has not been pursued by them. As our main result we have proved the existence of fixtures of TTC maps under conditions similar to those referred to above for TC maps. We have also shown the existence of fixtures under certain conditions of a different nature on f such as 'f has a convergent sequence of iterates' or 'there is a point p such that every neighbourhood of p contains a point x and its image fx'.

Just as in Banach's contraction theorem the sequence of iterates of a function starting from any point of the space, converges to the unique fixed point, in the case of a TTC map f we have shown that if f has neither a fixed point nor a fixed line then it has atmost one fixed plane P, and the sequence of iterates $\{f^n w\}$ converges to P for any w in the space. Next we have discussed the consequences of the existence of more than one fixture. For example, we have shown that if f has

exactly one fixed point and one fixed line then its fixed planes, if any, pass through either the fixed point or the fixed line.

The third chapter is devoted to the study of triangle contractive maps in general strictly convex and smooth normed linear spaces. Here we need define the area of a triangle. The most natural way of defining the area of a triangle formed by x,y,z would have been in terms of the lengths of the sides and perimeter of the triangle, as $[s(s-a)(s-b)(s-c)]^{1/2}$, where $a = ||y-z||$, $b = ||z-x||$, $c = ||x-y||$ and $2s = a+b+c$, but proceeding with this definition we do not get interesting results. We have taken an alternative formula for the area, namely half the product of a side and the shortest distance (or perpendicular distance) of the opposite vertex from it. To obtain the shortest distance we use the Birkhoff-James orthogonality (Definition 0.2.6(i)). In this case the area varies with the choice of the side and opposite vertex, hence we define the area as the triplet,

$$\Delta_1(x,y,z) = \{A(x;y,z),\ A(y;z,x),\ A(z;x,y)\}$$

where $A(x;y,z) = \frac{1}{2}$ (length of base vector through y and z) $\times$ (the shortest distance of x from the base)

If λ is the unique number (by strict convexity) such that

$$x-(\lambda y + (1-\lambda)z) \perp y-z,$$

then the shortest distance of x from the base $= ||x-(\lambda y+(1-\lambda)z)||$ and $A(x;y,z) = \frac{1}{2}\,||y-z||\ ||x-(\lambda y+(1-\lambda)z)||$.

We have one more similar way of defining the area of a triangle as the triplet

$$\Delta_2(x,y,z) = \{B(x;y,z),\ B(y;z,x),\ B(z;x,y)\}$$

where $B(x;y,z) = \frac{1}{2}$ (length of base vector joining y and z)

X (length of a vector joining x and a point in the base so that the base vector is orthogonal to this vector)

$$= \frac{1}{2}\ ||y-z||\ ||x-(\mu y + (1-\mu)z)||$$

with $y-z \perp x-(\mu y + (1-\mu)z)$; here μ is unique since the space is smooth.

An extension of the concept of inner product in a normed linear space is generalized inner product, which gives rise to yet one more definition of the area of a triangle as a sextuplet

$$\Delta_3(x,y,z) = \{C(x;y,z),\ C(x;z,y),\ C(y;z,x),$$

$$C(y;x,z),\ C(z;x,y),\ C(z;y,x)\}$$

where $C(x;y,z) = \frac{1}{2}\ ||x-y||\ ||z-y||\ \{1 - \langle x\hat{-}y,\ z\hat{-}y\rangle^2\}^{1/2}$.

Here $\langle x\hat{-}y, z\hat{-}y\rangle$ is the generalized inner product (Definition 0.2.9).

It is obvious that if the space is an inner product space then $\Delta_1 = \Delta_2 = \Delta_3$ = a unique number. Conversely, we have shown that if any two of the A's in Δ_1 or any two of the B's in Δ_2 are equal then orthogonality is symmetric, and hence if dimension is greater than two then it is an inner product space. Also if in any normed linear space any two of the C's in Δ_3 are

equal then the space is an inner product space irrespective of the dimension. For a two-dimensional space in which orthogonality is symmetric, we have proved that the areas Δ_1 and Δ_2 of the triangle with vertices (x_1, x_2), (y_1, y_2) and (z_1, z_2) are each equal to $\frac{1}{2} |(x_1 y_2 - x_2 y_1) + (y_1 z_2 - y_2 z_1) + (z_1 x_2 - z_2 x_1)|$ where coordinates are with respect to two orthogonal vectors. This expression for the area is same as that already known for an Euclidean plane. Thus we have obtained a characterization of a normed linear space X in which the orthogonality is symmetric. This adds to the list of characterizations already known, for example, orthogonality is symmetric if and only if (i) the rectangular constant m(X) (Definition 0.2.13) is $\sqrt{2}$ (Joly [20]), or (ii) the Lipschitz constant k(X) of radial retraction (Definition 0.2.15) is one (de Figueiredo and Karlovitz [14]), or (iii) the metric projection bound MPB(X) (Definition 0.2.16) is one (Smith [28]).

In the second part of third chapter we have defined triangle contractive maps of three types corresponding to the three areas Δ_1, Δ_2 and Δ_3. It is shown thatfor such triangle contractive maps (i) absence of continuity implies existence of fixtures and (ii) there exists a fixed direction such that lines parallel to it are mapped into lines parallel to it. Using these two results we have established the existence of fixtures for the three types of maps on a normed linear space, under fairly general conditions.

There are several geometric characterizations of inner product spaces amongst normed linear spaces. In the third chapter some such characterizations arose out of our consideration of different forms of areas of a triangle. In the fourth chapter we give certain refinements of some of the well known characterizations. It is known that in a normed linear space X if $x \perp y$ implies $x \perp_i y$ then X is an inner product space. In our main result we have proved that it is enough to have the above implication for unit vectors. Using this some known characterizations due to Day [5,6], Kapoor and Prasad [22] and Holub [18], (Theorem 0.2.8(iv)) have been improved upon. As another corollary we have proved that if $||x|| = ||y|| = 1$ and $x \perp y$ implies $||x+y||^2 + ||x-y||^2 = 2\,[||x||^2+||y||^2]$ then the space is inner product if orthogonality is assumed to be symmetric. But sans symmetry we could only prove that the space is strictly convex.

The following two characterizations of uniformly non-square Banach spaces (Definition 0.2.4(iv)) appear in the literature :

(i) A Banach space X is uniformly non-square if and only if the Lipschitz constant k(X) of radial retraction is less than two (Thele [31])

(ii) A Banach space X is uniformly non-square if and only if its metric projection bound MPB(X) is less than two (Smith [28]).

Looking at these results, and the characterizations of symmetry of orthogonality given by de Figueireds and Karlovitz and by

Smith mentioned earlier, we felt that the two numbers namely the Lipschitz constant $k(X)$ of radial retraction and the metric projection bound $MPB(X)$ may be equal. In the fifth chapter we have substantiated our feeling by proving the equality. Next we have tried to obtain the values of metric projection bound for the space $\ell_p^n(MPB(\ell_p^n))$, and have succeeded in obtaining a formula for $MPB(\ell_p^2)$, thus giving rise to some interesting questions involving ℓ_p norms.

0.2. In this section we present some basic definitions, notations and theorems which will be needed in the thesis.

For all normed linear spaces discussed in this thesis the scalars are assumed to be real and their field is denoted by R. S denotes the unit sphere in the normed linear space and θ its null element, $B_r(x)$ denotes a closed ball with centre at x and radius r. The dual of space X is denoted by X^*.

Definition 0.2.1. H is a Hilbert space ,

(i) For x, y ε H, $L(x,y) = \{\alpha x + (1-\alpha)y : \alpha \varepsilon R\}$ is called a line through x and y.

(ii) If $f : H \to H$ is such that for some $\alpha > 0$ and for any x,y,z ε H, either $\Delta(fx,fy,fz) \leq \alpha\, \Delta(x,y,z)$ or $||fx-fy|| \leq \alpha||x-y||$, $||fy-fz|| \leq \alpha||y-z||$ and $||fz-f\,x|| \leq \alpha||z-x||$ then f is called triangle expansion bounded (TEB). Here if $0 < \alpha < 1$ then f is called triangle contractive (TC).

(iii) A map $f : H \to H$ is called <u>generalized triangle expansion bounded</u> (GTEB) if there exists $h > 0$ such that for any $x, y, z \in H$, either

$$\Delta(fx, fy, fz) \leq h \max \{\Delta(x,y,z), \Delta(fx,fy,z), \tfrac{1}{2}[\Delta(x,fy,z) + \Delta(fx,y,z)]\}$$

or

$$||fx-fy|| \leq h \max \{||x-y||, ||x-fx||, ||y-fy||, \tfrac{1}{2}[||x-fy||+||y-fx||]\},$$

$$||fy-fz|| \leq h \max \{||y-z||, ||y-fy||, ||z-fz||, \tfrac{1}{2}[||y-fz|| + ||z-fy||]\}$$

and $||fz-fx|| \leq h \max \{||z-x||, ||z-fz||, ||x-fx||, \tfrac{1}{2}[||z-fx||, ||x-fz||]\}.$

If f is GTEB with $0 < h < 1$ then it is called <u>generalized triangle contractive</u> (GTC).

(iv) A TEB or a TC map is said to have a <u>fixture</u> if there is a point z with $fz = z$ (a fixed point) or a line L such that $fL \subset L$ (a fixed line).

In the following theorems Daykin and Dugdale [10] have given various sufficient conditions for a TEB or a TC map f to have a fixture.

<u>Theorem 0.2.2(i)</u> If f is TEB but not continuous then fH is a part of a fixed line.

Theorem 0.2.2(ii). If f is TEB and has a convergent sequence of iterates then it has a fixture.

Theorem 0.2.2(iii). If f is TC and there is a sequence $\{x_n\}$ of points in a finite-dimensional space H with $||x_n - fx_n|| \to 0$ then f has a fixture.

Theorem 0.2.2(iv). If H is finite-dimensional, f is TC and has a sequence of iterates which converges to a line then f has a fixture.

Theorem 0.2.2(v). If H is finite-dimensional, f is TC and $\{x_n\}$ is a sequence of iterates such that $\lim\inf ||x_n - x_{n+1}||$ is finite and $\Delta(x_n, x_{n+1}, x_{n+2}) \to 0$, then f has a fixture.

Definition 0.2.3. H is a Hilbert space,

(i) For $x, y, z \in H$, $P(x,y,z) = \{\alpha x + \beta y + (1-\alpha-\beta)z : \alpha, \beta \in R\}$ is called a plane through x, y and z.

(ii) If $f : H \to H$ is such that for some $\alpha > 0$ and for any $x_1, x_2, x_3, x_4 \in H$, either

$$V(fx_1, fx_2, fx_3, fx_4) \leq \alpha\, V(x_1, x_2, x_3, x_4)$$

(where $V(x_1, x_2, x_3, x_4)$ denotes the volume of the tetrahedron with vertices at x_1, x_2, x_3, x_4)

or $\Delta(fx_i, fx_j, fx_k) \leq \alpha\, \Delta(x_i, x_j, x_k)$ for all i, j and k

or $||fx_i - fx_j|| \leq \alpha ||x_i - x_j||$ for all i and j,

then f is called tetrahedron expansion bounded (TTEB), and is called tetrahedron contractive (TTC) if $0 < \alpha < 1$.

(iii) A TTEB or a TTC map f is said to have a fixture if there is a $z \varepsilon H$ with $fz = z$ or a line L with $fL \subset L$ or there is a plane P with $fP \subset P$ (fixed plane).

In the third chapter and subsequently we have used the symbols $||x||$ and $q(x)$ alternatively for the facility of writing the Gâteaux derivative of the norm. The <u>right</u> (left) <u>Gâteaux derivative</u> of norm functional in the direction of y is

$$q'_+(x,y) = \lim_{t \to 0+} \frac{||x+ty|| - ||x||}{t}$$

$$\left(q'_-(x,y) = \lim_{t \to 0-} \frac{||x+ty|| - ||x||}{t}\right)$$

The above two limits exist because q is a convex functional. If

$$q'_+(x,y) = q'_-(x,y)$$

we write the common value as $q'(x,y)$ and call it the <u>Gâteaux derivative</u> of the norm at x in the direction of y.

The different types of normed linear spaces considered by us are defined below :

<u>Definition 0.2.4.</u> X is a normed linear space,

(i) X is said to be <u>uniformly convex</u> if and only if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $||\frac{x+y}{2}|| \leq 1 - \delta(\varepsilon)$ whenever $||x-y|| \geq \varepsilon$ and $||x|| = ||y|| = 1$.

(ii) X is called <u>strictly convex</u> if and only if $||x+y|| = ||x|| + ||y||$ implies $x = ty$, $t > 0$ whenever $x \neq \theta$ and $y \neq \theta$.

(iii) X is said to be smooth if and only if its norm is Gâteaux differentiable at every point of the unit sphere S.

(iv) X is called uniformly non-square if and only if there is a positive number δ such that there do not exist elements x and y of S for which $\|\frac{x+y}{2}\| > 1 - \delta$ and $\|\frac{x-y}{2}\| > 1-\delta$.

Definition 0.2.5. If X is a smooth space, then the normalized duality map is defined as $J : X \to X^*$, (where for $x \in X$ we put $J(x) = J_x \in X^*$) such that $J_{\lambda x} = \lambda J_x$, $J_x(x) = \|x\|^2$ and $\|J_x\| = \|x\|$. For any $y \in X$ we put $J_x(y) = (J_x, y)$.

Orthogonality and inner product spaces

The natural definition of orthogonality relation between the elements of an inner product space is that $x \perp y$ if and only if the inner product (x,y) is zero. In a normed linear space X other notions of orthogonality have been given which are equivalent to this in case the norm arises from an inner product. We describe them here.

Definition 0.2.6(i). Birkhoff-James orthogonality : x is orthogonal to y in the sense of Birkhoff-James ($x \perp y$) if and only if

$$\|x+\lambda y\| \geq \|x\| \text{ for all } \lambda \in R.$$

This definition was used by Birkhoff [2] and by Fortet [15, 16]. Later on James [19] studied this orthogonality relating it to other concepts such as strict convexity, smoothness, weak compactness, linear functionals and hyperplanes.

Since we will be mainly using this orthogonality, by x orthogonal to y (or $x \perp y$) we will mean orthogonal in Birkhoff-James sense.

(ii) Isosceles orthogonality : x is orthogonal to y in the isosceles sense ($x \perp_i y$) if and only if $||x+y|| = ||x-y||$.

(iii) Pythagorean orthogonality : x is orthogonal to y in the Pythagorean sense ($x \perp_p y$) if and only if $||x+y||^2 = ||x||^2 + ||y||^2$.

It is well-known that orthogonality is homogeneous i.e. $x \perp y$ implies $\alpha x \perp \beta y$ for all $\alpha, \beta \in R$; but not symmetric in general i.e. $x \perp y$ does not necessarily imply $y \perp x$. The orthogonality is called right (left) unique if for each $x \neq \Theta$ and $y \in X$, there exists only one α such that $x \perp \alpha x + y$ ($\alpha x+y \perp x$); it is called right (left) additive if

$$x \perp y \text{ and } x \perp z \implies x \perp y+z$$

$$(y \perp x \text{ and } z \perp x \implies y+z \perp x).$$

The following facts about orthogonality are also well-known:

Theorem 0.2.7(i). Let X be a normed linear space, $x \neq \Theta$ and $y \in X$. Then there exists a number α such that $x \perp \alpha x+y$.

Theorem 0.2.7(ii). $x \perp y \implies (J_x, y) = 0$, where J is the normalised duality map , and X is smooth

Theorem 0.2.7(iii). (James [19]) In a normed linear space orthogonality is right (left) unique if and only if the space is smooth (strictly convex).

Theorem 0.2.7(iv) (Sundaresan [29]) In a two dimensional space there exist non-zero vectors x and y such that $x \perp y$ and $x+y \perp x-y$.

We shall be referring to the following characterizations of inner product spaces :

Theorem 0.2.8(i) (Jordan and von Neumann [21]) A normed linear space is an inner product space if for every pair $x, y \in X$,

$$||x+y||^2 + ||x-y||^2 = 2\left[||x||^2 + ||y||^2\right]$$

i.e. in any parallelogram the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the sides (The parallelogram law).

Day [5] has improved this result in the following :

Theorem 0.2.8(ii) A normed linear space is an inner product space if $||x+y||^2 + ||x-y||^2 = 4$ whenever $||x|| = ||y|| = 1$.

Theorem 0.2.8(iii) (Birkhoff [2], James [19], Day [5]) Let X be a normed linear space of dimension greater than two. X is an inner product space if and only if the orthogonality is symmetric.

Theorem 0.2.8(iv) (Day [4,5], Kapoor and Prasad [22], Holub [18]) For a normed linear space X the following are equivalent :

(a) X is an inner product space

(b) $x \perp_i y \implies x \perp_p y$

(c) $x \perp_p y \implies x \perp_i y$

(d) $x \perp_p y \Longrightarrow x \perp y$

(e) $x \perp y \Longrightarrow x \perp_p y$

(f) $x \perp_i y \Longrightarrow x \perp y$

(g) $x \perp y \Longrightarrow x \perp_i y$

<u>Generalized inner product</u>

<u>Definition 0.2.9</u>. The generalized inner product $\langle x,y\rangle$ of x with y is defined to be the right Gâteaux derivative of the convex functional $f(x) = \frac{1}{2}||x||^2$, at x in the direction of y. Thus

$$\langle x,y\rangle = f'_+(x,y) = ||x||\ q'_+(x,y).$$

Tapia [30], Laugwitz [23], and Prasad [25] have given different proofs of the fact that the space must be inner product whenever generalized inner product $\langle x,y\rangle$ is linear in x or symmetric.

It is also well-known that if X is a smooth normed linear space, then

$$q'_+(x,y) = \frac{(J_x,y)}{||x||}$$

hence $\langle x,y\rangle = (J_x,y)$ and then $x \perp y \Longleftrightarrow \langle x,y\rangle = 0$.

<u>k-set contractions and related concepts</u>

In chapters I and II we have studied triangle contractive and tetrahedron contractive maps on an infinite-dimensional Hilbert space. For drawing significant conclusions we have to impose certain conditions on such maps. This is done using

the following definitions and results which we more or less reproduce from [24].

In all that follows X is a real Banach space, and D a bounded subset of X in Definition 0.2.10.

Definition 0.2.10(i) The set measure of non-compactness of D is defined as

$$\gamma(D) = \text{Inf}\,\{d > 0 : D \text{ can be covered by a finite number of sets of diameter} \leq d\}$$

It follows that $\gamma(D) = \gamma(\bar{D})$, $\gamma(\lambda D) = |\lambda|\,\gamma(D)$, $\gamma(D) \leq \gamma(Q)$ whenever $D \subset Q$ with Q bounded, $\gamma(D \cup Q) = \max\{\gamma(D), \gamma(Q)\}$ and $\gamma(D) = 0$ if and only if $\bar{D}$ is compact.

Closely associated with γ is the concept of a k-set-contraction $T : G \to X$ defined to be a bounded continuous map such that $\gamma(T(D)) \leq k\,\gamma(D)$ for each bounded $D \subset G$ and some constant $k \geq 0$. A bounded continuous map $T : G \to X$ is set-condensing (or densifying) if $\gamma(T(D)) < \gamma(D)$ for each bounded $D \subset G$ with $\gamma(D) \neq 0$. It follows that every k-set-contractive map with $k < 1$ is set condensing and that every set condensing map is 1-set-contractive (but the reverse implications do not hold).

(ii) Analogous to the concept of set measure of non-compactness is that of ball measure of non-compactness of D defined by

$$\chi(D) = \text{Inf}\,\{r > 0 : D \text{ can be covered by a finite number of balls with centres in X and radius } r\}.$$

The above mentioned properties of γ are also valid for χ. As in the case of γ, corresponding to χ we have k-ball-contractions and ball-condensing maps.

For fixed point theory generally the same argument works for maps $T : \bar{D} \to X$, defined either in terms of γ or in terms of χ.

Apart from the above we also need the following notions :

Definition 0.2.11(i) Let $D \subset X$ and let $T : \bar{D} \to X$ such that $||Tx-Ty|| \leq \alpha||x-y||$ for $x,y \in D$ and some $\alpha > 0$. Then T is called contractive if $\alpha < 1$, and non-expansive if $\alpha = 1$.

(ii) If $T : D \to X$ be such that

$$||x-y|| \leq ||(1+r)(x-y) - r(Tx-Ty)||,$$

for all $x,y \in D$ and all $r > 0$, then T is called pseudo-contractive mapping.

This class of maps is more general than the class of non-expansive maps, and it has the useful property [4] that T is pseudo-contractive if and only if $(I-T)$ is monotone i.e. $(J_{x-y}, (I-T)(x-y)) \geq 0$ for all $x, y \in D$, where J is the normalized duality map.

(iii) A continuous mapping $T : \bar{D} \to X$ is called locally almost non-expansive (lane) if given any $x \in \bar{D}$ and $\varepsilon > 0$, there exists a weak neighbourhood N_x of x in $\bar{D}$ (depending also on ε) for which

$$||Tx'-Ty|| \leq ||x'-y|| + \varepsilon, \text{ for all } x',y \in N_x.$$

Before passing on to the relevant results we recall that $T : D \subset X \to X$ is <u>compact</u> if T is continuous and $\overline{T(A)}$ is compact whenever A is bounded. T is <u>completely continuous</u> (or <u>strongly continuous</u>) if for any sequence $\{x_n\}$ in D such that $x_n \rightharpoonup x_o$ in D, $Tx_n \to Tx_o$ in X as $n \to \infty$.

<u>Theorem 0.2.12</u>(i) Let X be a real Banach space, D a bounded open subset of X and T either a 1-set-contractive or 1-ball-contractive map of $\bar{D}$ into X for which the following condition holds :

(a) 'There exists $x_o \in D$ such that if $Tx-x_o = \alpha(x-x_o)$ for some $x \in \partial D$ then $\alpha \leq 1$, where ∂D denotes the boundary of D'.

Then T has a fixed point in $\bar{D}$ if and only if T satisfies the following condition :

(b) 'If $\{x_n\}$ is any sequence in $\bar{D}$ such that $x_n-Tx_n \to 0$ as $n \to \infty$ then there exists $x' \in \bar{D}$ with $x'-Tx' = 0$'.

<u>Remark</u> In case T is set-condensing or ball-condensing (and, in particular, k-set or k-ball contraction with $0 \leq k < 1$), the sufficiency part of Theorem 0.2.12(i) remains valid without the assumption of condition (b), for the latter is always true for this class of mappings.

<u>Theorem 0.2.12</u>(ii) Let X be a uniformly convex Banach space, D a bounded open convex subset of X, S a non-expansive map of $\bar{D}$ into X, and C a completely continuous map of $\bar{D}$ into X. If the mapping $T = S+C : \bar{D} \to X$ satisfies the condition (a) of Theorem 0.2.12(i), then T has a fixed point in $\bar{D}$.

Theorem 0.2.12(iii) Let X be a uniformly convex Banach space, D a bounded open convex subset of X, L a 'lane' mapping of $\bar{D}$ into X, and C a completely continuous mapping of $\bar{D}$ into X. If the mapping $T = L+C : \bar{D} \to X$ satisfies (a) of Theorem 0.2.12(i) on ∂D, then T has a fixed point in $\bar{D}$.

Theorem 0.2.12(iv) Let X be a uniformly convex Banach space and let T be a mapping of X into X which is both k-set-contractive for some $k \geq 0$ and pseudo-contractive. If there exists a bounded open convex subset D of X with θ in D such that $Tx \neq \lambda x$ for $x \in \partial D$ and $\lambda > 1$, then T has a fixed point in X.

The following theorem occurs in [13] and [26].

Theorem 0.2.12(v) Let H be a Hilbert space and let $T : H \to H$ be a continuous and either set-condensing or ball-condensing function such that for some $r > 0$, $||x|| = r$ implies that $Tx \neq mx$ for any $m > 1$. Then T has a fixed point in the closed ball of centre θ and radius r.

Remark. We have essentially used the results of Theorems 0.2.12(i) - (iv) in the Hilbert space setting, where D is a closed ball. Such results appear elsewhere also, but [24] is a survey type paper and gives us the required results at one place.

Rectangular constant, Lipschitz constant and Metric Projection Bound.

We now give some definitions and results which have been mainly used in Chapter IV and V.

Definition 0.2.13. The rectangular constant of a normed linear space X is defined as the number

$$m(X) = \sup_{x \perp y} \frac{||x|| + ||y||}{||x+y||} .$$

Joly [20] has shown that $\sqrt{2} \leq m(X) \leq 3$ and $m(X) = \sqrt{2}$ implies the symmetry of orthogonality in X, and hence if dimension $X \geq 3$ then the space is inner product. del Rio and Benitez [11] have proved the following

Theorem 0.2.14. X is an inner product space if and only if $m(X) = \sqrt{2}$.

Definition 0.2.15. The Lipschitz constant k(X) of a normed linear space X is the infimum of all numbers k for which

$$||Tx-Ty|| \leq k\ ||x-y|| \text{ for all } x,y \in X.$$

Here T is the radial retraction on the unit ball, defined by

$$\begin{aligned} Tx &= x \quad \text{if} \quad ||x|| \leq 1 \\ &= \frac{x}{||x||} \text{ if } ||x|| \geq 1 \end{aligned}$$

Definition 0.2.16. If M is a non-trivial closed proper subspace of a normed linear space X, then the set of best approximations to x from M is defined by

$$P_M(x) = \{y \in M : ||x-y|| = d(x,M)\}$$

where $d(x,M) = \text{Inf}\ \{||x-y|| : y \in M\}$. The subspace M is called proximinal if $P_M(x)$ contains atleast one point for every $x \in X$.

If M is proximinal the norm of P_M is defined by

$$||P_M|| = \text{Sup } \{||y|| : y \in P_M(x), \ ||x|| \leq 1\}.$$

Now the <u>metric projection bound</u> of X written as MPB(X) is defined to be

$$MPB(X) = \text{Sup } \{||P_M|| : M \text{ proximinal subspace of } X\}.$$

CHAPTER I

TRIANGLE CONTRACTIVE MAPS IN A HILBERT SPACE

1.1. Introduction. Daykin and Dugdale conjectured in [10] that fixtures of triangle contractive (TC) maps in finite-dimensional Hilbert spaces exist. Daykin himself proved the conjecture for a two-dimensional space in [9], and later on Ang and Hoa, in [1], established its truth for any finite-dimensional space. In the main result of this chapter we have proved the existence of fixtures of TC maps in the infinite-dimensional case under fairly general conditions, and thus extending some other results of Daykin and Dugdale [10] regarding fixtures, for all Hilbert spaces.

1.2. Preliminaries. Let H be a real Hilbert space. For $x,y,z \in H$ we denote by $\Delta(x,y,z)$ the area of the triangle formed by x,y and z. Let $L(y,z) = \{\alpha y+(1-\alpha)z : \alpha \in R\}$ be the line through y and z. If $u,v \in L(y,z)$ then it can be easily seen that $L(u,v) = L(y,z)$. By $\pi(x,L(y,z))$ we denote the distance of x from $L(y,z)$. From the geometry of Hilbert space we have

$$\pi(x,L(y,z)) = \|x-y\| \left[1-(x\hat{-}y,y\hat{-}z)^2\right]^{1/2}$$

where $x\hat{-}y$ and $y\hat{-}z$ denote unit vectors in the direction of x-y and y-z. Now we can give an explicit formulation for the area of a triangle as

$$\Delta(x,y,z) = \frac{1}{2}\, ||y-z||\ \pi(x,\ L(y,z))$$

$$= \frac{1}{2}\, ||y-z||\ ||x-y||\ [1-(x\hat{-}y, y\hat{-}z)^2]^{1/2}.$$

It can be easily verified that

$$||x-y||\ ||y-z||\ [1-(x\hat{-}y, y\hat{-}z)^2]^{1/2}$$

$$= ||y-z||\ ||z-x||\ [1-(y\hat{-}z, z\hat{-}x)^2]^{1/2}$$

$$= ||z-x||\ ||x-y||\ [1-(z\hat{-}x, x\hat{-}y)^2]^{1/2}$$

Hence the area of a triangle is a fixed number.

A map $f : H \to H$ is called triangle expansion bounded (TEB) if there exists an $\alpha > C$ such that for any $x,y,z \in H$ either $\Delta(fx,fy,fz) \leq \alpha\, \Delta(x,y,z)$, or $||fx-fy|| \leq \alpha\, ||x-y||$, $||fy-fz|| \leq \alpha\, ||y-z||$ and $||fz-fx|| \leq \alpha\, ||z-x||$. If in the above definition $0 < \alpha < 1$ then we say that f is triangle contractive (TC).

Definition 1.2.1. A self map f of a normed linear space X is said to have the property (*) if it satisfies the following condition :

'Either f has a fixed point or there exists a sequence $\{(\lambda_n, x_n)\}$ in $R \times X$ such that $||x_n|| \to \infty$, $\lambda_n > 1$ and $fx_n = \lambda_n x_n$'.

It is known for example, that a map will have the property (*) if it has an asymptotic bifurcation point $\lambda > 1$. We recall that $\lambda \in R$ is called an asymptotic bifurcation point of a map $f : X \to X$ if there exists a sequence $\{(\lambda_n, x_n)\}$ in $R \times X$ such that $\lambda_n \to \lambda$, $||x_n|| \to \infty$ and $fx_n = \lambda_n x_n$ [17].

The lemma given below lists sufficient conditions on f to have the property (*).

Lemma 1.2.1. Let f be a continuous map of a Hilbert space into itself. Each of the following is a sufficient condition on f to have the property (*) :

(i) f is set-condensing or ball-condensing

(ii) $f = g+h$, where $g : H \to H$ is non-expansive and $h : H \to H$ is completely continuous.

(iii) f is locally almost non-expansive (lane) or $f = g+h$ where $g : H \to H$ is lane and h is completely continuous.

(iv) f is both pseudo-contractive and k-set contractive for some $k \geq 0$.

(v) f is k-set-contractive and I-f is monotone.

(vi) f is 1-ball-contractive or 1-set-contractive and there exists a sequence $\{x_n\}$ in H such that $||x_n - fx_n|| \to 0$ as $n \to \infty$.

Proof. We assume in each case that f has no fixed point, and then get the required conclusion by referring to Theorems 0.2.12(i) - (v).

(i) Since f is set-condensing or ball-condensing and has no fixed point, taking $r = n$ (a positive integer) in Theorem 0.2.12(v) we get the sequence $\{(\lambda_n, x_n)\}$ with $||x_n|| = n$ and $fx_n = \lambda_n x_n$ where $\lambda_n > 1$.

(ii) If in Theorem 0.2.12(ii) we take $D = \{x : ||x|| < n\}$, n is a positive integer, we get the sequence $\{(\lambda_n, x_n)\}$ as above.

(iii) We proceed in the same way as in (ii) with reference to the Theorem 0.2.12(iii).

(iv) Theorem 0.2.12(iv) gives the desired result if we proceed as in (ii).

(v) It is known [4] that I-f is monotone if and only if f is pseudo-contractive. Hence (v) is equivalent to (iv).

(vi) Here f satisfies the hypothesis of Theorem 0.2.12(i), hence proceeding as in (ii) we get the sequence $\{(\lambda_n, x_n)\}$ as required.

1.3. The Main result. The following theorem along with Lemma 1.2.1 proves our main result that a large class of TC maps have fixtures.

Theorem 1.3.1. Let $f : H \to H$ be a TC map having the property (*) then f has either a fixed point or a fixed line.

Proof. If f is not continuous then we know (Theorem 0.2.2(i)) that there exists a line L such that $fH \subset L$. So we may assume that f is continuous. If f has no fixed point then we have for each n a $\lambda_n > 1$ and x_n with $||x_n|| \to \infty$ and $fx_n = \lambda_n x_n$. We may assume that $\hat{x}_n = \dfrac{x_n}{||x_n||}$ converges weakly to an element $w \in H$, otherwise we pass on to such a subsequence, clearly $||w|| \leq 1$. For any $x \in H$ and for any $\beta \in (\alpha, 1)$

$$\begin{aligned} ||fx_n - fx|| &\geq ||fx_n|| - ||fx|| = ||x_n - x|| - ||x_n - x|| + \lambda_n ||x_n|| - ||fx|| \\ &\geq ||x_n - x|| + (\lambda_n - 1)||x_n|| - (||fx|| + ||x||) \\ &\geq \beta ||x_n - x||, \text{ when n is sufficiently large.} \end{aligned}$$

Since f is TC, we have for sufficiently large n and $x,y \in H$,

$$2\Delta(fx,fy,fx_n) = ||fx-fy|| \ ||fx-fx_n||[1-(fx\hat{-}fy,fx\hat{-}fx_n)^2]^{1/2}$$

$$\leq \alpha \ 2\Delta(x,y,x_n) = \alpha||x-y|| \ ||x-x_n||[1-(x\hat{-}y,x\hat{-}x_n)^2]^{1/2}$$

From the above two inequalities we get for $x,y \in H$ and for sufficiently large n,

(1) $$||fx-fy||[1-(fx\hat{-}fy,fx\hat{-}fx_n)^2]^{1/2} \leq \frac{\alpha}{\beta}||x-y||[1-(x\hat{-}y,x\hat{-}x_n)^2]^{1/2}$$

Observing that $\frac{x_n}{||x_n||}$, $\frac{-x+x_n}{||-x+x_n||}$ and $\frac{-fx+fx_n}{||-fx+fx_n||}$. Converge weakly to w, and that β can be arbitrarily close to one, we obtain from (1),

(2) $$||fx-fy||[1-(fx\hat{-}fy,w)^2]^{1/2} \leq \alpha||x-y||[1-(x\hat{-}y,w)^2]^{1/2}.$$

Now taking $y = x_n$, with n large enough we get

$$||fx-fx_n|| \ [1-(fx\hat{-}fx_n,w)^2]^{1/2} \leq \alpha||x-x_n|| \ [1-(x\hat{-}x_n,w)^2]^{1/2},$$

hence $w \neq 0$. Taking limit as $n \to \infty$, $||w|| = 1$.

Let $M = [w]^{\perp}$, the subspace orthogonal to w. Let P_M denote the orthogonal projection onto M. Since $||P_Mx-P_My|| = ||x-y|| \ [1-(x\hat{-}y,w)^2]^{1/2}$, the inequality (2) yields

$$||P_Mfx-P_Mfy|| \leq \alpha||P_Mx-P_My|| \leq \alpha||x-y||$$

and then by Banach contraction principal there is a fixed point of P_Mf. Let $P_Mfz = z$, and L be the line through z parallel to w. Take any $z' \in L$, $z' \neq z$, then $\frac{z'-z}{||z'-z||} = w$, and the inequality (2) gives

$$||P_M fz - P_M fz'|| = ||fz-fz'|| \, [1-(fz \hat{-} fz', w)^2]^{1/2}$$

$$\leq \alpha ||z-z'|| \, [1-(z \hat{-} z', w)^2]^{1/2} = 0$$

Hence $P_M fz = P_M fz' = z$. Thus $fz' \in L$, and this shows that L is fixed.

Corollary (Ang and Hoa [1]) In a finite-dimensional Hilbert space all TC maps have fixtures.

Proof. A continuous TC map in a finite-dimensional space has the property (*), and hence the proof follows from Theorem 1.3.1.

1.4. Some more result. We now take up the extensions of some of the results of Daykin and Dugdale [10] to the infinite-dimensional case. The following theorem being an extension of Theorem 0.2.2.(iii).

Theorem 1.4.1. If $f : H \to H$ is TC and there exists a sequence $\{x_n\}$ in H such that $||x_n - fx_n|| \to 0$ then f has a fixture.

We first prove two lemmas which we need to prove the theorem

Lemma 1.4.2. Let $f : H \to H$ be triangle contractive. If either f is not continuous or is not bounded (does not take bounded sets into bounded sets) then fH is contained in a line.

Proof. If f is discontinuous then it is known (Theorem 0.2.2(i)) that fH is contained in a line.

Suppose $\{x_n\}$ is a bounded sequence such that $||fx_n|| \to \infty$. Take any $x, y \in H$ such that $fx \neq fy$. Then for all sufficiently large n we must have

$$\|fx-fy\|\ \|fx-fx_n\|\ [1-(fx\hat{-}fy, fx\hat{-}fx_n)^2]^{1/2}$$

$$\leq \alpha\|x-y\|\ \|x-x_n\|\ [1-(x\hat{-}y, x\hat{-}x_n)^2]^{1/2}$$

$$\leq \alpha\|x-y\|\ \|x-x_n\|.$$

Since $\|x-x_n\|$ is bounded and $\|fx-fx_n\| \to \infty$, we must have $(fx\hat{-}fy, fx\hat{-}fx_n) \to 1$ as $n \to \infty$. Assuming, without loss of generality, that $fx\hat{-}fx_n$ converges weakly to w, we have

$$(fx\hat{-}fy, w) = 1, \text{ therefore } \|w\| = 1$$

and $fy-fx = \lambda w$ i.e. $fy = fx + \lambda w$.

This proves that fH is contained in the line through fx and parallel to w.

Lemma 1.4.3. Let f be a bounded continuous TC map on H. Then f is 1-ball-contractive.

Proof. Let D be any bounded subset of H. Let $r > 0$ and $x_1, \ldots, x_n \in H$ such that $D \subset \bigcup_{i=1}^{n} B_r(x_i)$. Put $D_i = D \cap B_r(x_i)$, $i = 1,2,\ldots,n$. Assume, without loss of generality, that $fD_i \subset B_r(fx_i)$, $i = 1,2,\ldots,m$, and $fD_i \not\subset B_r(fx_i)$ for $i=m+1,\ldots,n$. Choose y_i such that $fy_i \in fD_i \sim B_r(fx_i)$, $i = m+1,\ldots,n$. Let $d_i = \sup_{y\in D_i} \|fx_i-fy\|$. Choose a positive integer λ_i so large that $[r^2 - (\frac{d_i}{\lambda_i})^2]^{1/2} > \alpha r$. Let $z_{ji} = fx_i + j(\frac{d_i}{\lambda_i})\, w_i$, $i=m+1,\ldots,n$, $-\lambda_i \leq j \leq \lambda_i$, where $w_i = \dfrac{fy_i-fx_i}{\|fy_i-fx_i\|}$.

Let $z \in D$ such that $fz \not\in (\bigcup_{i=1}^{m} B_r(fx_i)) \cup (\bigcup_{i=m+1}^{n} B_r(z_{ji}))$, $-\lambda_i \leq j \leq \lambda_i$, then $z \in D_i$ for some i where $m+1 \leq i \leq n$, and for such an i,

$$||fx_i-fy_i|| > r > ||x_i-y_i||,$$

hence we must have

$$\frac{1}{2}\alpha r^2 < \frac{1}{2}\alpha r\,||fx_i-fy_i|| < \frac{1}{2}\left[r^2-\left(\frac{d_i}{\lambda_i}\right)^2\right]^{1/2}||fx_i-fy_i||$$

$$\leq \Delta(fx_i,fy_i,fz) \leq \alpha\,\Delta(x_i,y_i,z)$$

$$\leq \frac{\alpha}{2}\,||x_i-y_i||\,r \leq \frac{1}{2}\alpha r^2$$

which is not possible. Thus fD can also be covered by a finite number of balls of radius r. Therefore $\chi(f(D)) \leq \chi(D)$, which was to be proved.

Proof of Theorem 1.4.1. If f is discontinuous or not bounded then Lemma 1.4.2 gives the result. If f is continuous and bounded then, by Lemma 1.4.3, f is 1-ball contractive and since $||x_n-fx_n|| \to 0$, from Lemma 1.2.1(vi) it follows that f has the property (*). Then Theorem 1.3.1 shows that f has a fixture.

The following theorem is the infinite-dimensional analogue of Theorem 0.2.2(iv), its proof runs parallel to the corresponding theorem but we give it for the sake of completeness.

Theorem 1.4.4. Let f be a TC map on H. Suppose that there exists a sequence of iterates of f converging to a line L in H, then f has a fixture.

Proof. Let $\{x_n\}$ be a sequence of iterates of f with $fx_n = x_{n+1}$, converging to the line L. All the points of the sequence are distinct, because otherwise f has a fixed point. If $\lim\inf ||x_n - x_{n+1}|| = 0$, then $\{x_n\}$ has a subsequence satisfying the condition of Theorem 1.4.1, and hence f has a fixture. We therefore, assume that $\lim\inf ||x_n - x_{n+1}|| > 0$. Also $||fx_{n-1} - fx_n|| > \alpha ||x_{n-1} - x_n||$ for infinitely many n. Assuming L is not a fixed line, $p \in L$ such that $fp \notin L$. We consider two cas

Case I $\{x_n\}$ **is bounded. Since** $\{x_n\}$ **is bounded and converges to the line L, it has a subsequence which can be renamed as** $\{x_n\}$ **converging to a point w of L.** fx_n **also converges (f is continuous), and it converges to** $fw \in L$**. If** $fw \neq w$ **then**

$$\Delta(fx_{n-1}, fx_n, f^p) \to \Delta(w, fw, fp) > 0$$

But $\Delta(x_{n-1}, x_n, p) \to 0$**, hence a choice of n is possible such that**

$$||fx_{n-1} - fx_n|| > \alpha ||x_{n-1} - x_n||$$

and $$\Delta(fx_{n-1}, fx_n, fp) > \alpha\, \Delta(x_{n-1}, x_n, p)$$

which contradicts that f is TC, and so L is fixed.

Case II. $\{x_n\}$ is unbounded. Choose $\varepsilon > 0$ very small such that $\varepsilon < \pi(fp, L)$, and choose an integer k such that $\pi(x_k, L) < \varepsilon$ and $\pi(fx_k, L) < \varepsilon$. Since $\{x_n\}$ is unbounded, we can get a subsequence $\{x_{n_i}\}$ such that $||x_k - x_{n_i}|| \to \infty$ as $i \to \infty$ and $||x_k - fx_{n_i}|| > ||x_k - x_{n_i}||$ for all i. Hence for sufficiently large i, $||fx_k - fx_{n_i}|| > \alpha ||x_k - x_{n_i}||$. Now for sufficiently large i, $\Delta(x_k, x_{n_i}, p)$ is approximately $\frac{1}{2}\, ||x_k - x_{n_i}||\, \pi(x_k, L) < \frac{1}{2} ||x_k - x_{n_i}||\, \varepsilon$,

and $\Delta(fx_k, fx_{n_i}, fp)$ is approximately

$$\frac{1}{2} \|fx_k - fx_{n_i}\| \pi(fp, L) > \frac{1}{2} \alpha \|x_k - x_{n_i}\| \pi(fp, L).$$

Thus we again get a contradiction that f is TC, hence L is fixed.

We have not been able to prove Theorem 0.2.2(v) fully for the infinite-dimensional case, but give below its partial extension.

Theorem 1.4.5. Let f be a TC map on H and $\{x_n\}$ is an unbounded sequence of iterates of f with $\liminf \|x_n - x_{n+1}\| = \lambda$. Then f has a fixture.

Proof. If $\liminf \|x_n - x_{n+1}\| = 0$ then there exists a subsequence of $\{x_n\}$ satisfying the hypothesis of Theorem 1.4.1, hence there is a fixture.

Assuming $\lambda > 0$, we get a subsequence $\{x_{n_i}\}$ which, without loss of generality, can again be taken as $\{x_n\}$ such that $\lim_{n \to \infty} \|x_n - fx_n\| = \lambda$. If necessary, we can further take a subsequence of $\{x_n\}$ such that $\hat{x}_n$ converges weakly to w, with $\lim \|x_n\| = \lim \|x_{n+1}\| = \infty$.

We have

$$\frac{\|x_{n+1}\| - \|x_{n+1} - x_n\|}{\|x_{n+1}\|} \le \frac{\|x_n\|}{\|x_{n+1}\|} \le \frac{\|x_{n+1}\| + \|x_n - x_{n+1}\|}{\|x_{n+1}\|},$$

hence $\lim \frac{\|x_n\|}{\|x_{n+1}\|} = 1$. For any $\beta \in (\alpha, 1)$ Choose $\delta > 0$ such that $\frac{1-\delta}{\beta(1+\delta)} > 1$.

For sufficiently large n, and for any x,

$$\frac{||fx-x_{n+1}||}{\beta||x-x_n||} = \frac{\dfrac{||fx-x_{n+1}||}{||x_{n+1}||}}{\beta\,\dfrac{||x-x_n||}{||x_{n+1}||}} \geq \frac{1-\delta}{\alpha(1+\delta)} > 1 \text{ , because}$$

$$\frac{||fx||}{||x_{n+1}||} \leq \delta \text{ and } \frac{||x||}{||x_{n+1}||} \leq \delta \text{ .}$$

Thus $||fx-fx_n|| > \beta||x-x_n||$ for sufficiently large n. Now proceeding as in Theorem 1.3.1 we conclude that f has a fixture.

1.5. An example. Many examples of TC maps have been given by Daykin and Dugdale in [10] for two-dimensional spaces. We give an example of a TEB map in an infinite-dimensional Hilbert space which has no fixture.

Let $H = \ell_2$, for $x = (x_1, x_2, \ldots) \in \ell_2$, define $fx = (1, \frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}, \ldots)$, then f is TEB and has no fixture.

CHAPTER II

TETRAHEDRON CONTRACTIVE MAPS IN A HILBERT SPACE

2.1. Introduction. We have seen in Chapter I that if triangles go down under a self map of a Hilbert space then fixtures of the map exist modulo certain conditions on the map. In this chapter we establish similar results when tetrahedrons go down.

2.2 Preliminaries. In a real Hilbert space H, we denote by $P(x,y,z)$ the plane through $x,y,z \in H$, where

$$P(x,y,z) = \{\alpha x+\beta y+\gamma z : \alpha+\beta+\gamma = 1, \alpha,\beta,\gamma \in R\}$$

It can be easily seen from the definition of $P(x,y,z)$ that if $u,v,w \in P(x,y,z)$ then $P(u,v,w) = P(x,y,z)$, and also if two planes intersect then they intersect in a line.

For $x,y,z,u \in H$, put $a = x-y$, $b = y-z$, $c = z-x$, $d = u-x$, $e = u-y$, $f = u-z$, then it can be easily verified that

$$(||b||^2||c||^2-(b,c)^2)||d||^2-(c,d)^2||b||^2-(d,b)^2||c||^2+$$
$$+2(b,c)(c,d)(d,b)$$
$$=(||c||^2||d||^2-(c,d)^2)||e||^2-(d,e)^2||c||^2-(e,c)^2||d||^2+$$
$$+2(c,d)(d,e)(e,c)$$
$$= \text{etc.}$$

The distance of u from $P(x,y,z)$ is

$$\rho(u,P(x,y,z)) = \begin{cases} ||d|| \text{ if } x = y = z \\ \pi(u,L) \text{ if } x,y,z \in L(\text{a line}) \\ 0 \text{ if } u \in P(x,y,z) \\ \frac{1}{2\Delta(x,y,z)} \lambda, \text{ otherwise} \end{cases}$$

here $\lambda^2 = (||b||^2||c||^2-(b,c)^2)||d||^2-(c,d)^2||b||^2-(d,b)^2||c||^2$

$+ 2(b,c)(c,d)(d,b)$.

If $V(u,x,y,z)$ denotes the volume of the tetrahedron formed by u,x,y and z, then

$$\begin{aligned} V(u,x,y,z) &= \frac{1}{3} \Delta(x,y,z)\ \rho(u,P(x,y,z)) \\ &= \frac{1}{6} [(||b||^2||c||^2-(b,c)^2)||d||^2-(c,d)^2||b||^2 \\ &\quad -(d,b)^2||c||^2+2(b,c)(c,d)(d,b)]^{1/2} \\ &= \frac{1}{6} ||b||\ ||c||\ ||d||\ [1-(\hat{b},\hat{c})^2-(\hat{c},\hat{d})^2-(\hat{d},\hat{b})^2 \\ &\quad + 2(\hat{b},\hat{c})(\hat{c},\hat{d})(\hat{d},\hat{b})]^{1/2}. \end{aligned}$$

A map $f : H \to H$ is called tetrahedron contractive (TTC) if for some α $0 < \alpha < 1$ and for any $x_1,x_2,x_3,x_4 \in H$ either

(i) $V(fx_1,fx_2,fx_3,fx_4) \leq \alpha\, V(x_1,x_2,x_3,x_4)$

or (ii) $(fx_i,fx_j,fx_k) \leq \alpha\, \Delta(x_i,x_j,x_k)$ for all i,j and k

or (iii) $||fx_i-fx_j|| \leq \alpha\, ||x_i-x_j||$ for all i and j.

If in the above definition α is any positive real number then f is called tetrahedron expansion bounded (TTEB).

2.3 Fixtures of TTC maps

Theorem 2.3.1. If f is TTEB but not continuous then fH is part of a fixed line or a fixed plane.

Proof. We assume that f is discontinuous at x, then there exists a sequence $\{x_n\}$ such that $x_n \to x$, andfor some $\varepsilon > 0$, $\|fx_n - fx\| \geq \varepsilon$, $n = 1,2,\ldots$. For any $y \in H$, $\Delta(y,x_n,x) \to 0$. If $\Delta(fy,fx_n,fx)$ also converges to zero for all $y \in H$, then taking $y = x_1$ and $L = L(fx_1,fx)$ we get that $\{fx_n\} \to L$, hence $fy \in L$ for all $y \in H$ and L is a fixed line containing fH.

If $\Delta(fy,fx_n,fx) \not\to 0$ for some $y = y_0 \in H$ then for some $\varepsilon > 0$ there exists as subsequence $\{fx_{n_i}\}$ of the sequence $\{fx_n\}$ such that $\Delta(fy_0,fx_{n_i},fx) \geq \varepsilon$, $i = 1,2,\ldots$. Since f is TTEB and for all $z \in H$, $V(z,y_0,x_{n_i},x) \to 0$, we have necessarily $V(fz,fy_0,fx_{n_i},fx) \to 0$. Choose $z = x_{n_1}$ and let $P = P(fx_{n_1},fy_0,fx)$, then

$$V(fx_{n_1},fy_0,fx_{n_i},fx) = \frac{1}{3}\,\Delta(fx_{n_1},fy_0,fx)\;\rho(fx_{n_i},P)$$
$$\geq \frac{1}{3}\,\varepsilon\;\rho(fx_{n_i},P).$$

Hence $\rho(fx_{n_i},P) \to 0$. We claim that $fH \subset P$. If not, let $y \in H$ with $fy \notin P$, that is $\rho(fy,P) > 0$. Since $\Delta(fy_0,fx_{n_i},fx) \geq \varepsilon$, and $\rho(fy,P(fy_0,fx_{n_i},fx)) \to \rho(fy,P) > 0$, hence $V(fy,fy_0,fx_{n_i},fx) \not\to 0$, a contradiction.

Corollary 2.3.2. If f is TTEB and has a convergent sequence of iterates then it has a fixture.

Proof. It follows easily from Theorem 2.3.1

Theorem 2.3.3. Let f be a TTC map satisfying any one of the conditions (i) to (v) listed in Lemma 1.2.1, then f has a fixture.

Proof. As in Theorem 1.3.1 we obtain a sequence $\{x_n\}$ in H and a sequence $\{\lambda_n\}$ of numbers such that $\lambda_n > 1$, $||x_n|| \to \infty$, $fx_n = \lambda_n x_n$, and for sufficiently large n and any $x \in H$, $\beta \in (\alpha, 1)$ $||fx_n - fx|| > \beta\, ||x_n - x||$, and the bounded sequence $\{\hat{x}_n\}$ converges weakly to w.

Since f is TTC, therefore, for any three points $x, y, z \in H$, we have for all sufficiently large n, either

$$\Delta(fx, fy, fx_n) \leq \alpha\, \Delta(x, y, x_n),$$

$$\Delta(fy, fz, fx_n) \leq \alpha\, \Delta(y, z, x_n),$$

$$\Delta(fz, fx, fx_n) \leq \alpha\, \Delta(z, x, x_n) \text{ and}$$

$$\Delta(fx, fy, fz) \leq \alpha\, \Delta(x, y, z)$$

or

$$V(fx, fy, fz, fx_n) \leq \alpha\, V(x, y, z, x_n).$$

Proceeding as in Theorem 1.3.1, we obtain from these inequalities either

$$(1)\quad \begin{aligned} ||fx-fy||\,[1-(fx\hat{-}fy, w)^2]^{1/2} &\leq \alpha ||x-y||\ [1-(x\hat{-}y, w)^2]^{1/2} \\ ||fy-fz||\,[1-(fy\hat{-}fz, w)^2]^{1/2} &\leq \alpha ||y-z||\ [1-(y\hat{-}z, w)^2]^{1/2} \\ ||fx-fz||\,[1-(fx\hat{-}fz, w)^2]^{1/2} &\leq \alpha ||x-z||\ [1-(x\hat{-}z, w)^2]^{1/2} \end{aligned}$$

or

$$(2)\quad \begin{aligned} &||fx-fy||^2||fx-fz||^2[1-(fx\hat{-}fy, fx\hat{-}fz)^2-(fx\hat{-}fy, w)^2 - \\ &-(fx\hat{-}fz, w)^2+2(fx\hat{-}fy, fx\hat{-}fz)(fx\hat{-}fy, w)(fx\hat{-}fz, w)] \\ &\leq \alpha^2||x-y||^2||x-z||^2[1-(x\hat{-}y, x\hat{-}z)^2-(x\hat{-}y, w)^2-(x\hat{-}z, w)^2 \\ &\qquad +2(x\hat{-}y, x\hat{-}z)(x\hat{-}y, w)(x\hat{-}z, w)] \end{aligned}$$

We put $y = x_n$ in (1) and (2). If (1) holds for infinitely many n's then in the limit the first inequality in (1) yields

$$(1-||w||^4)^{1/2} \leq \alpha(1-||w||^4)^{1/2} \text{ and so } ||w|| = 1.$$

On the other hand if (2) holds for infinitely many n's, we shall obtain in the limit

$$\begin{aligned} &||fx-fz||^2[1-(f\hat{x}-fz,w)^2-||w||^4-(f\hat{x}-fz,w)^2+2(f\hat{x}-fz,w)^2||w||^2] \\ (3) \quad &\leq \alpha^2||x-z||^2[1-(x\hat{-}z,w)^2-||w||^4-(x\hat{-}z,w)^2+2(x\hat{-}z,w)^2||w||^2]. \end{aligned}$$

Again taking $z = x_n$ in (3) we shall obtain the limit as

$$(1-||w||^4-2||w||^4+2||w||^6) \leq \alpha^2(1-||w||^4-2||w||^4+2||w||^6)$$

which again implies that $||w|| = 1$.

Hence $\{\hat{x}_n\}$ converges to w in the norm.

Let P_M be the orthogonal projection on the subspace $M = [w]^{\perp}$. Since $||w|| = 1$, from (1) and (2) we get that for any $x,y,z \in H$, either

$$\begin{aligned} &||P_M fx-P_M fy|| \leq \alpha||P_M x-P_M y|| \leq \alpha||x-y|| \\ (4) \quad &||P_M fy-P_M fz|| \leq \alpha||P_M y-P_M z|| \leq \alpha||y-z|| \\ &||P_M fz-P_M fx|| \leq \alpha||P_M z-P_M x|| \leq \alpha||z-x|| \end{aligned}$$

or

$$(5) \quad \Delta(P_M fx,P_M fy,P_M fz) \leq \alpha\, \Delta(P_M x,P_M y,P_M z) \leq \alpha\, \Delta(x,y,z)$$

The inequalities in (4) can be obtained by using the relation $||P_M x-P_M y|| = ||x-y||\,[1-(x\hat{-}y,w)^2]^{1/2}$, and (5) can be obtained as follows :

$2\Delta(P_M fx, P_M fy, P_M fz)$

$= ||P_M fx - P_M fy||\ ||P_M fx - P_M fz||\,[1-(P_M fx \hat{-} P_M fy, P_M fx \hat{-} P_M fz)^2]^{1/2}$

$= ||fx-fy||\ ||fx-fz||\,[1-(fx\hat{-}fy,w)^2]^{1/2}[1-(fx\hat{-}fz,w)^2]^{1/2}$

$[1-(P_M fx \hat{-} P_M fy, P_M fx \hat{-} P_M fz)^2]^{1/2}$

$= ||fx-fy||\ ||fx-fz||\,[1-(fx\hat{-}fy,w)^2-(fx\hat{-}fz,w)^2 +$

$(fx\hat{-}fy,w)^2(fx\hat{-}fz,w)^2-(P_M(fx\hat{-}fy),P_M(fx\hat{-}fx))^2]^{1/2}.$

Since $P_M(fx\hat{-}fy) = fx\hat{-}fy-(fx\hat{-}fy,w)w$, etc., we have

$(P_M(fx\hat{-}fy),P_M(fx\hat{-}fz))^2 = [(fx\hat{-}fy,fx\hat{-}fz)-(fx\hat{-}fy,w)(fx\hat{-}fz,w)]^2$

hence

$2\Delta(P_M fx, P_M fy, P_M fz)$

$= ||fx-fy||\ ||fx-fz||\,[1-(fx\hat{-}fy,w)^2-(fx\hat{-}fz,w)^2-(fx\hat{-}fy,fx\hat{-}fz)^2$

$+ 2(fx\hat{-}fy,w)(fx\hat{-}fz,w)(fx\hat{-}fy,fx\hat{-}fz)]^{1/2}$

$\leq 2\alpha\ \Delta(P_M x, P_M y, P_M z) \leq 2\alpha\ \Delta(x,y,z).$

From (4) and (5) we conclude that $P_M f$ is TC. It will have the property (*), since f has this property and P_M is non-expansive. Hence $P_M f$ will have a fixed point p or a fixed line L (Thm. 1.3.1).

In case $P_M f$ has a fixed point p, let the line through p and parallel to w be L′. Take any $y \in H$ and $z \in L'$, then either

$||P_M fp - P_M fy|| \leq \alpha||P_M p - P_M y||,$

$||P_M fp - P_M fz|| \leq \alpha||P_M p - P_M z||$ and

$||P_M fy - P_M fz|| \leq \alpha||P_M y - P_M z||$

or

$$\Delta(P_M fp,\ P_M fy,\ P_M fz) \le \alpha\ \Delta(P_M p,\ P_M y,\ P_M z).$$

If $fz \notin L'$ then both these conditions are violated, hence L' must be a fixed line of f.

In case L is a fixed line of $P_M f$ then let P be the plane through L parallel to w. Take any $z \in P$ with $x = P_M z$ and $y \in L$. If $fz \notin P$ then both the above conditions are again violated. Hence P must be a fixed plane of f.

Corollary 2.3.4. In a finite-dimensional Hilbert space every continuous TTC map will have the property (*) and hence will have a fixture.

Remark 2.3.5. We feel that all continuous and bounded TTC maps are 1-ball-contractions, and probably have fixtures.

Remark 2.3.6. It will be interesting to know under what conditions on f and/or M, the composite map $P_M f$ will have the property (*), if f has the property (*).

2.4 Existence of fixtures under different conditions.

Here we consider various sufficient conditions on TTEB or TTC maps (continuous or not) to have fixtures.

Theorem 2.4.1. Suppose f is TTEB and p is a point such that every neighbourhood of p contains a point x and its image fx, then either p is a fixed point or $L(p,fp)$ is a fixed line or there exists a fixed plane containing $L(p,fp)$.

Proof. We can choose a sequence $\{x_n\}$ such that $x_n \to p$ and $fx_n \to p$. Suppose p is not a fixed point and $L(p,fp)$ is not a fixed line. Let $z \in L(p,fp)$ with $fz \not\in L(p,fp)$. We claim that plane $P(p,z,fz)$ is a fixed plane. If not, take $u \in P(p,z,fz)$ where $fu \not\in P(p,z,fz)$. We observe that for sufficiently large n

$$||fx_n - fp|| > 0 \text{ but } ||x_n - p|| \to 0$$

and $\Delta(x_n,p,z) \to 0$ but $\Delta(fx_n,fp,fz) \to \Delta(p,fp,fz) > 0$. Also $V(u,x_n,p,z) \to 0$ but $V(fu,fx_n,fp,fz) \to V(fu,p,fp,fz) > 0$, and this leads to a contradiction.

Here it is also easy to see that $fH \subset P(p,z,fz)$.

Theorem 2.4.2. Suppose f is TTC and L is a line in H such that there exist sequences $\{u_n\} \subset L$ and $\{x_n\} \subset H$ with $||u_n - x_n|| < \frac{1}{n}$ and $||u_n - fx_n|| < \frac{1}{n}$ for all n. Then either L is a fixed line, or a plane containing L is a fixed plane.

Proof. $\{u_n\}$ is bounded then it has an accumulation point, hence by Theorem 2.4.1, f has a fixture.

Suppose $\{u_n\}$ is unbounded then $\{x_n\}$ is also unbounded. For sufficiently large m and k,

$$||fx_m - fx_k|| \geq ||x_m - x_k|| - ||fx_m - u_m|| - ||u_m - x_m|| - ||x_k - u_k|| - ||u_k - fx_k||$$
$$> \alpha\, ||x_m - x_k||.$$

Assuming L is not a fixed line, take $w \in L$ where $fw \not\in L$. We observe that approximately

$$\Delta(w,x_m,x_k) = \frac{1}{2}\, ||x_m - x_k||\ \pi(x_m,L) \leq \frac{1}{2}\, ||x_m - x_k||\ \frac{1}{m}$$

where m and k are sufficiently large, and

$$\Delta(fw,fx_m,fx_k) = \frac{1}{2}\,||fx_m-fx_k||\;\pi(fw,L)$$

$$> \frac{1}{2}\,\alpha||x_m-x_k||\,\frac{1}{m}$$

$$> \alpha\,\Delta(w,x_m,x_k).$$

Since f is TTC we must necessarily have

$$V(fz,fw,fx_m,fx_k) \leq \alpha\,V(z,w,x_m,x_k) \text{ for all } z \in H.$$

Let P be the plane through fw and L and take $z \in P$ such that $fz \notin P$, then

$$V(fz,fw,fx_m,fx_k) = \frac{1}{3}\,\Delta(fw,fx_k,fx_m)\;\rho(fz,P)$$

$$> \alpha\,V(z,w,x_k,x_m)$$

a contradiction. Hence P is a fixed plane.

Theorem 2.4.3. Let f be a TTC map and $\{x_n\}$ is a sequence converging to x with $||x_n-fx_n|| \to \lambda > 0$, then

(i) all the accumulation points of $\{fx_n\}$ are coplaner of which no three are collinear, and

(ii) if p,q,r are any three accumulation points of $\{fx_n\}$ then P(p,q,r) is a fixed plane containing fH.

Proof. Suppose p_1,p_2,p_3 and p_4 are four distinct accumulation points of $\{fx_n\}$. Since

$$||x-p_i|| \leq ||x-x_n||+||x_n-fx_n||+||fx_n-p_i|| \to \lambda,\ i = 1,2,3,4$$

and $||x-p_i|| \geq \lambda-\varepsilon$, $\varepsilon > 0$, hence $||x-p_i|| = \lambda$ for all i. Thus p_i,p_j,p_k are not collinear for any i,j and k. If

$p_4 \notin P(p_1,p_2,p_3)$ then $V(p_1,p_2,p_3,p_4) > 0$. We can find integers n_i,n_j,n_k,n_ℓ so that $V(fx_{n_i},fx_{n_j},fx_{n_k},fx_{n_\ell})$ is arbitrarily close to $V(p_1,p_2,p_3,p_4)$; but $V(x_{n_i},x_{n_j},x_{n_k},x_{n_\ell}) \to 0$. Also by the same token

$$\Delta(fx_{n_i},fx_{n_j},fx_{n_k}) > \alpha\, \Delta(x_{n_i},x_{n_j},x_{n_k})$$

and $||fx_{n_i}-fx_{n_j}|| > \alpha\, ||x_{n_i}-x_{n_j}||$,

this contradicts that f is TTC. Hence p_1,p_2,p_3 and p_4 are coplaner.

Let $P = P(p,q,r)$, take any $z \in H$ where $fz \notin P$, then for a suitable choice of integers n_i,n_j and n_k, the condition that f is TTC is again violated for the points $z,x_{n_i},x_{n_j},x_{n_k}$. Hence P is a fixed plane containing fH.

<u>Theorem 2.4.4</u>. Let f be TTC, and $\{x_n\}$ is a sequence converging to x with $||x_n-fx_n|| \to \lambda > 0$, $V(x_n,fx_n,f^2x_n,f^3x_n) \to 0$ and $\Delta(fx_n,f^2x_n,f^3x_n) \geq \mu > 0$ for all n. Then f has either a fixed line or a fixed plane.

<u>Proof</u>. If p,q,r are three accumulation points of $\{fx_n\}$ then, by Theorem 2.4.3, P(p,q,r) is a fixed plane containing fH. Morevoer $x \in P(p,q,r)$, because $fx_n,f^2x_n,f^3x_n \in fH \subset P(p,q,r) = P$, and by hypothesis

$$V(x_n,fx_n,f^2x_n,f^3x_n) = \frac{1}{3}\,\Delta(fx_n,f^2x_n,f^3x_n)\,\rho(x_n,P) \to 0$$

and $\Delta(fx_n,f^2x_n,f^3x_n) \geq \mu > 0$.

If $\{fx_n\}$ has only two accumulation points p and q then either $L(p,q)$ is a fixed line or $P(fw,p,q)$ is a fixed plane where $w \in L(p,q)$ but $fw \notin L(p,q)$. If not, let $u \in P(fw,p,q)$ such that $fu \notin P(fw,p,q)$, hence $V(fu,fw,p,q) > 0$. We can choose integers n_i and n_j so that $V(fu,fw,fx_{n_i},fx_{n_j}) > 0$ but $V(u,w,x_{n_i},x_{n_j}) \to 0$, moreover $\Delta(fw,fx_{n_i},fx_{n_j}) > 0$, but $\Delta(w,x_{n_i},x_{n_j}) \to 0$ and $||fx_{n_i}-fx_{n_j}|| > \alpha\, ||x_{n_i}-x_{n_j}||$. This is impossible since f is TTC.

Finally if $\{fx_n\}$ has only one limit point p then $L(x,p)$ is a fixed line otherwise $P(fw,x,p)$ is a fixed plane where $fw \notin L(x,p)$ for some $w \in L(x,p)$.

Theorem 2.4.5. If f is TTC and x,y,z are distinct points of H with $fx = y$, $fy = z$ and $fz = x$, then the plane $P(x,y,z)$ is fixed.

Proof. Suppose $w \in P(x,y,z)$ but $fw \notin P(x,y,z)$, then $V(w,x,y,z)=0$ but $V(fw,fx,fy,fz) = V(fw,x,y,z) > 0$ and $\Delta(fx,fy,fz) = \Delta(y,z,x) > \alpha\, \Delta(x,y,z)$. Since f is TTC we must have

$$||fx-fy|| \leq \alpha\, ||x-y||, ||fy-fz|| \leq \alpha ||y-z||, ||fz-fx|| \leq ||z-x|| \text{ etc.}$$

This implies that $||y-z|| \leq \alpha^3 ||y-z||$, hence $x = y = z$, this contradicts the hypothesis.

Theorem 2.4.6. If f is TTC and P is a plane with non-collinear $x,y,z \in P$ such that $fx,fy,fz \in P$ and $||fx-fy|| \geq \beta||x-y|| > 0$ and $\Delta(fx,fy,fz) \geq \beta\, \Delta(x,y,z)$ where $\beta > \alpha$, then P is a fixed plane. Further $f^n w \to P$ for all $w \in H$.

Proof. For any $w \in H$, and $x,y,z \in P$, we have by hypothesis,

(1) $$V(fw,fx,fy,fz) \leq \alpha\, V(w,x,y,z).$$

If $w \in P$ then $V(w,x,y,z) = 0$, hence $fw \in P(fx,fy,fz) = P(x,y,z)$, and thus P is fixed.

Since the inequality (1) holds for all $w \in H$, we have for each positive integer n,

$\frac{1}{3}\,\Delta(fx,fy,fz)\ \rho(f^{n+1}w,P) \leq \alpha\,\frac{1}{3}\,\Delta(x,y,z)\ \rho(f^{n}w,P)$, hence

$$\rho(f^{n+1}w,P) \leq \frac{\alpha}{\beta}\ (f^{n}w,P) \leq \ldots \leq (\frac{\alpha}{\beta})^{n}\ \rho(fw,P).$$

Since $\beta > \alpha$, $(\frac{\alpha}{\beta})^{n} \to 0$ as $n \to \infty$ giving that $\rho(f^{n+1}w,P) \to 0$, consequently $f^{n}w \to P$.

2.5 Consequences of existence of more than one fixture

Proposition 2.5.1. If f is TTC and p,q,r,s are fixed points of f, then either they are collinear or coplaner.

Proof. Assuming $p \neq q$, $||fp-fq|| = ||p-q|| > \alpha||p-q||$, hence either (i) or (ii) of the definition of a TTC map is satisfied for the points p,q,r and s.

If (i) is not satisfied, then

$\Delta(fp,fq,fr) = \Delta(p,q,r) \leq \alpha\,\Delta(p,q,r)$, which implies $\Delta(p,q,r) = 0$, and similarly $\Delta(q,r,s) = \Delta(r,s,p) = \Delta(s,p,q) = 0$. Thus p,q,r and s are collinear.

If (i) is satisfied then $V(p,q,r,s) = 0$, hence the points are coplaner.

<u>Proposition 2.5.2.</u> If f is TTC and x,y,z are its fixed points then either they lie on a line or a fixed plane P. In the latter case $f^n w \to P$ for all $w \in H$.

<u>Proof.</u> If x,y and z are not collinear then the result follows from Theorem 2.4.6, because $||fx-fy|| = ||x-y|| \geq \beta\ ||x-y||$ and $\Delta(fx,fy,fz) = \Delta(x,y,z) \geq \beta\ \Delta(x,y,z)$ where $\alpha < \beta < 1$.

<u>Proposition 2.5.3.</u> If f is TTC, P_1 and P_2 are two planes and $x_i,y_i,z_i \in P_i$, $i = 1,2$, such that (i) x_i,y_i,z_i are non-collinear, (ii) $fx_i,fy_i,fz_i \in P_i$, (iii) $||fx_i-fy_i|| \geq \beta||x_i-y_i||$ and (iv) $\Delta(fx_i,fy_i,fz_i) \geq \beta\ \Delta(x_i,y_i,z_i)$, where $\beta > \alpha$. Then f has a fixed line L-the intersection of P_1 and P_2. Also $f^n w \to L$ for all $w \in H$. Further if M is any other fixed line then f has one and only one fixed point p - the intersection of L and M.

<u>Proof.</u> P_1 and P_2 both satisfy the hypothesis of Theorem 2.4.6, hence both are fixed planes and $f^n w \to P_i$, $i = 1,2$, for each $w \in H$. Therefore, P_1 and P_2 intersect in a fixed line L and $f^n w \to L$ for all $w \in H$. If M is also a fixed line then $f^n w \to L$ for all $w \in M$, hence L and M intersect in a fixed point p. Clearly there can not be any other fixed point.

<u>Proposition 2.5.4.</u> Let f be TTC and P and Q its distinct fixed planes then each has a fixture (either a fixed point or a fixed line). In case of each having a fixed line, the lines may be coincident.

Proof. We consider two cases :

Case I. Suppose f is TC on P as well as on Q, then f has a fixture in P and also in Q.

Case II. Suppose f is not TC on atleast one of them, say P. Then it satisfies the hypothesis of Theorem 2.4.6 on P, hence $f^n w \to P$ for all $w \in H$, and in particular for all $w \in Q$. Since Q is a fixed plane, P and Q intersect in a fixed line L and in this case the two fixed lines coincide.

Proposition 2.5.5. If f is TTC with a fixed plane P and either a fixed point $x \notin P$ or a fixed line L not intersecting P, then f is TC on P (and so has a fixture in P).

Proof. If f is not TC on P, then hypothesis of Theorem 2.4.6 is satisfied on P, and we arrive at a contradiction that $p = f^n p \to P$ in the first case, or $f^n w \to P$ for all $w \in L$ in the second case.

Proposition 2.5.6. Let f be TTC then limits of convergent sequences of iterates (whenever they exist) are coplaner.

Proof. Suppose $f^n x_i \to y_i$ as $n \to \infty$, $i = 1,2,3,4$. Put $V = V(y_1, y_2, y_3, y_4)$, $\Delta = \Delta(y_1, y_2, y_3)$, $\sigma = \|y_1 - y_2\|$ and $V_n = V(f^n x_1, f^n x_2, f^n x_3, f^n x_4)$, $\Delta_n = \Delta(f^n x_1, f^n x_2, f^n x_3)$ and $\sigma_n = \|f^n x_1 - f^n x_2\|$, $n = 1,2,\ldots$.

Clearly, $V_n \to V$, $\Delta_n \to \Delta$ and $\sigma_n \to \sigma$. If $V \neq 0$ then $\Delta \neq 0$ and hence $\sigma \neq 0$, but from this it follows that, for sufficiently

large n,

$$\sigma_{n+1} > \alpha\, \sigma_n,\ \Delta_{n+1} > \alpha\, \Delta_n \text{ and } V_{n+1} > \alpha\, V_n,$$

which contradicts the TTC property of f. Hence V = 0, which implies that y_1, y_2, y_3 and y_4 are coplaner.

From the results proved above the following theorem can be proved easily :

Theorem 2.5.7 : For a TTC map f the following hold :

(i) If two different fixed lines of f meet at a point p then p is a fixed point of f or if two fixed planes intersect in a line L then L is a fixed line.

(ii) If f has no fixed point and no fixed line then it has atmost one fixed plane P and if so then $f^n w \to P$ for all $w \in H$.

(iii) If f has exactly one fixed point p and one fixed line L, then its fixed planes, if any, pass through either the fixed point p or intersect the fixed line L.

(iv) If f has three or more non-collinear fixed points then all lie on a fixed plane P. Moreover any other fixed line or fixed plane will intersect P and $f^n w \to P$ for all $w \in H$.

Proof (i) Trivial.

(ii) It is not possible for f to have two fixed planes, because then, by Proposition 2.5.4, f will also have fixed lines or fixed points or both.

Thus if f has a fixed plane P, then f can not be TC on P, therefore, f satisfies the hypothesis of Theorem 2.4.6 on P which implies that $f^n w \to P$ for all $w \in H$.

(iii) Suppose P is a fixed plane and it neither passes through p nor it intersects L, then Proposition 2.5.5 implies that f is TC on P and hence f has a fixed point or a fixed line in P - a contradiction.

(iv) The first part and that $f^n w \to P$ for all $w \in H$ follows from Proposition 2.5.2, and then any other fixed line or fixed plane will intersect P.

2.6 Example of a TTC map which is not TC

Let $H = R^3$ and define $f : R^3 \to R^3$, where $f(u,v,w) = (u+1, v+1, \frac{w}{2})$, then f is non-expansive, TEB and TTC but not TC.

CHAPTER III

AREA OF A TRIANGLE AND TC MAPS IN A NORMED LINEAR SPACE

3.1 Introduction. In this chapter we would see how far we can extend the results of the first chapter to TC maps and their fixtures in a normed linear space. But for pursuing that we want to have with us a notion of the area of a triangle in a general normed linear space. This is what we do in the first half of this chapter and in so doing obtain certain interesting results characterizing inner product spaces. In the second half we discuss the existence of fixtures under different conditions, and implications of existence of more than one fixture.

In Chapter I the area of a triangle in an inner product space had a clear cut meaning - the Euclidean area, which equals half the product of a side and the perpendicular distance from the opposite vertex, or which is given by a host of other formulae. In a general normed linear space X one obvious defining formula for the area of a triangle x,y,z would be

$$\Delta_o(x,y,z) = [s(s-a)(s-b)(s-c)]^{1/2}$$

where $s = \frac{1}{2}(a+b+c)$, $a = ||y-z||$, $b = ||z-x||$, $c = ||x-y||$. But this definition of the area of a triangle does not work well as far as the existence of fixtures of TC maps is concerned.

We shall, therefore, adopt the formula - half the product of length of a side and the shortest distance of opposite vertex from it. To obtain this shortest distance (or the perpendicular distance) of opposite vertex from a side we use the Birkhoff-James orthogonality or the generalized inner product (Definitions 0.2.6(i) and 0.2.9). In either case the shortest distance depends on the choice of a particular side and opposite vertex, hence we have taken the liberty of assigning a triplet or a sextuplet of numbers as the area of a triangle, as against the usual assignment of just one number as the area. We may remark here that these definitions of the area of a triangle do not possess many of the properties of the area of a triangle which hold good in the case of an inner product space, for example, if we take a point on one of the sides of a given triangle and consider two triangles with one common side joining this point to the opposite vertex then the sum of the areas of these two triangles may not be equal to the area of the given triangle.

Here for the purpose of clarity we forego some generality and we assume the normed linear space X to be strictly convex and smooth.

3.2 Area of type I. When we employ the Birkhoff-James orthogonality to obtain the shortest distance of the vertex from the opposite side we get the area of a triangle as a triplet, which we call the area of type I given by

$$\Delta_1(x,y,z) = \{A(x;y,z),\ A(y;z,x),\ A(z;x,y)\},\quad x,y,z \in X,$$

where $A(x;y,z) = \frac{1}{2} ||y-z|| \times$ shortest distance of the vertex x from the line through the points y and z

$$= \frac{1}{2} ||y-z|| \; ||x-(\lambda y+(1-\lambda)z)||$$

Here λ is the unique number (by strict convexity) such that

$$x - (\lambda y + (1-\lambda)z) \perp y-z$$

Similarly $A(y;z,x)$ and $A(z;x,y)$ can also be expressed.

From the above definition the following proposition is obvious :

Proposition 3.2.1. For any $x,y,z \in X$,

(i) $A(x;y,z) = \frac{1}{2} ||y-z|| \, (J_{x-(\lambda y+(1-\lambda)z)}, x-z)^{1/2}$

where J is the normalized duality map (Definition 0.2.5) and $(J_{x-(\lambda y+(1-\lambda)z)}, y-z) = 0$.

(ii) $A(x;y,z) = A(x+w;y+w,z+w)$ for any $w \in X$.

(iii) $A(x;y,z)$ is a continuous function of x,y and z.

We give our first result characterizing an inner product space.

Lemma 3.2.2. If dimension $X \geq 3$ and for all $x,y,z \in X$, $A(x;y,z) = A(y;z,x)$, then X is an inner product space.

Proof. Let $u,v \in X$ with $u \perp v$, then $A(u;v,\theta) = \frac{1}{2}||v|| \; ||u||$ and $A(v;\theta,u) = \frac{1}{2} ||u|| \; ||v-\lambda u||$ where $v-\lambda u \perp u$. Now, $A(u;v,\theta) = A(v;\theta,u)$ implies $||v|| = ||v-\lambda u||$ and so $||v|| \leq ||v+ku||$ for all k, hence $v \perp u$. Thus orthogonality is symmetric and consequently the space is inner product.

3.3 Area of type II. In giving the area of type I we take the shortest distance of the opposite vertex from a side as the norm of the vector joining the vertex and a point in the side so that the vector is orthogonal to the side. Turning the other way round we can take the shortest(!)distance as the length of a vector joining the vertex to a point in the side so that the side is orthogonal to the vector, and thus we get the area of type II given by

$$A_2(x,y,z) = \{B(x;y,z),\ B(y;z,x),\ B(z;x,y)\},\ x,y,z \in X,$$

where $B(x;y,z) = \frac{1}{2}\ ||y-z||\ ||x-(\mu y+(1-\mu)z)||$, μ is the unique number (by smoothness) such that

$$y-z \perp x-(\mu y+(1-\mu)z).$$

Similarly $B(y;z,x)$ and $B(z;x,y)$ can also be expressed.

Since orthogonality is not symmetric, $B(x;y,z) \neq A(x;y,z)$, in general.

The following obvious proposition is again there.

Proposition 3.3.1. For $x,y,z \in X$,

(i) $B(x;y,z) = \frac{1}{2}\ ||y-z||\ ||x-z- \dfrac{(J_{y-z},x-z)}{||y-z||^2}\ (y-z)||$

where J is the normalized duality map and

$(J_{y-z}, x-(\mu y+(1-\mu)z)) = 0.$

(ii) $B(x;y,z) = B(x+w;y+w,z+w)$ for any $w \in X$.

(iii) $B(x;y,z)$ is a continuous function of x,y and z.

Another characterizing property of an inner product space is given by the following :

Lemma 3.3.2. Suppose dimension $X \geq 3$ and for all $x, y, z \in X$, $B(x;y,z) = B(y;z,x)$, then X is an inner product space.

Proof. Let u and v be such that $u \perp v$, then $B(u;v,\theta) = \frac{1}{2} ||v|| \, ||u-\mu v||$ where $\mu = (J_v,u)/||v||^2$, and $B(v;\theta,u) = \frac{1}{2}||u|| \, ||v||$. $u \perp v$ implies that $||u|| \leq ||u+kv||$ for all k, and by hypothesis $||u|| = ||u-\mu v||$, hence $u-\mu v \perp v$, which is impossible unless $\mu = 0$ and therefore $v \perp u$. Symmetry of orthogonality shows that the space is inner product.

3.4 A characterization of symmetry of orthogonality

From the proofs of Lemmas 3.2.2 and 3.3.2 we note that if $A(x;y,z) = A(y;z,x)$ or $B(x;y,z) = B(y;z,x)$ then the orthogonality is symmetric. Conversely, if dimension $X \geq 3$ and orthogonality is symmetric then X is an inner product space, hence $A(x;y,z) = A(y;z,x)$ and $B(x;y,z) = B(y;z,x)$. If dimension $X = 2$, then we have the following :

Lemma 3.4.1. If dimension $X = 2$ and orthogonality is symmetric then $B(x;y,\theta) = B(y;\theta,x)$ for any $x,y \in X$.

Proof. Since $B(x;y,\theta) = \frac{1}{2} ||y|| \, ||x - \frac{(J_y,x)}{||y||^2} y||$

$$= \frac{1}{2} ||y|| \, ||x|| \, ||\hat{x} - (J_{\hat{y}},\hat{x})\hat{y}||$$

and $B(y;\theta,x) = \frac{1}{2} ||x|| \, ||y|| \, ||\hat{y} - (J_{\hat{x}},\hat{y}) \hat{x}||$, it is sufficient to prove that

$$\|\hat{x} - (J_{\hat{y}},\hat{x})\hat{y}\| = \|\hat{y} - (J_{\hat{x}},\hat{y})\hat{x}\|.$$

Introducing the coordinate system with respect to any two orthogonal vectors - as done by Day in [5, p. 330] - we let

$\hat{x} = (x_1, x_2)$, $\hat{y} = (y_1, y_2)$ and $J_{\hat{y}} = (p_y, q_y)$ so that

$$p_y y_1 + q_y y_2 = 1 = \sup_{\xi,\eta \neq 0} \frac{|p_y \xi + q_y \eta|}{\|(\xi,\eta)\|}$$

$$\begin{aligned}
\|\hat{x}-(J_{\hat{y}},\hat{x})\hat{y}\| &= \|(x_1,x_2)-(p_y x_1+q_y x_2)(y_1,y_2)\| \\
&= \|(x_1-p_y x_1 y_1-q_y x_2 y_1, x_2-p_y x_1 y_2-q_y x_2 y_2)\| \\
&= \|(q_y y_2 x_1-q_y x_2 y_1, p_y y_1 x_2-p_y x_1 y_2)\| \\
&= |x_1 y_2-x_2 y_1| \, \|(q_y,-p_y)\|.
\end{aligned}$$

But $(J_{\hat{y}},(q_y,-p_y)) = p_y q_y - p_y q_y = 0$.

Therefore, we must have, Day [5], that

$$1 = |p_y y_1+q_y y_2| = \|(y_1,y_2)\| \, \|(q_y,-p_y)\| = \|(q_y,-p_y)\|$$

Thus $\|\hat{x}-(J_{\hat{y}},\hat{x})\hat{y}\| = |x_1 y_2-x_2 y_1|$. Similarly $\|\hat{y}-(J_{\hat{x}},\hat{y})\hat{x}\| = |x_1 y_2-x_2 y_1|$, and the desired result now easily follows.

<u>Theorem 3.4.2.</u> The orthogonality in a normed linear space is symmetric if and only if either $A(x;y,z) = A(y;z,x)$ or $B(x;y,z) = B(y;z,x)$ for all $x, y, z \in X$.

<u>Proof</u>. If the orthogonality is symmetric then we observe that

$$A(x;y,z) = B(x;y,z) \text{ and } A(y;z,x) = B(y;z,x)$$

Also $B(x;y,z) = B(x-z;y-z,\theta)$

$$= B(y-z;\theta,x-z), \text{ by Lemma 3.4.1}$$

$$= B(y;z,x).$$

Hence from this the result follows.

The converse follows from the proofs of Lemmas 3.2.2 and 3.3.2.

<u>Theorem 3.4.3</u>. If in a two-dimensional space the orthogonality is symmetric then

$$\Delta_1(x,y,z) = \Delta_2(x,y,z) = \frac{1}{2}\,|(y_1z_2-y_2z_1)+(z_1x_2-z_2x_1)+(x_1y_2-x_2y_1)|$$

where $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $z = (z_1,z_2)$ with respect to two orthogonal unit vectors.

<u>Proof</u>. From Lemma 3.4.1 and Theorem 3.4.2 it follows that $\Delta_1(x,y,z) = \Delta_2(x,y,z) = B(x;y,z)$. Also from the proof of Lemma 3.4.1 we have

$$B(x;y,z) = B(x-z;y-z,\theta)$$

$$= \frac{1}{2}\,|(x_1-z_1)(y_2-z_2)-(x_2-z_2)(y_1-z_1)|$$

$$= \frac{1}{2}\,|(y_1z_2-y_2z_1)+(z_1x_2-z_2x_1)+(x_1y_2-x_2y_1)|.$$

This completes the proof.

<u>Remark</u>. The result of Theorem 3.4.3 is interesting in the sense that here we get the same expression for the area of a triangle as that already known for a triangle in an Euclidean plane.

3.5 Example. We consider the following examples which show that in a general normed linear space, triangles may exist with (i) all the numbers of triplets for Δ_1 or Δ_2 different, or (ii) two of the numbers equal but third different. In these examples we also observe that all the three numbers are equal only in the trivial case (i.e. when each of the numbers is zero).

For details of computations in the following examples where $X = \ell_3^2$, one can refer to the examples of TC maps in §3.12.

(i) If $x = (2,1)$, $y = (1,1)$ and $\theta = (0,0)$ then

$$A(x;y,\theta) = \frac{1}{2^{4/3}}\ ,\ A(\theta;x,y) = \frac{1}{2} \text{ and } A(y;\theta,x) = \frac{9^{1/3}}{2(1+2\sqrt{2})^{2/3}}$$

(ii) If $x = (1,0)$, $y = (0,1)$ and $\theta = (0,0)$ then

$$A(x;y,\theta) = \frac{1}{2} = A(y;\theta,x) \text{ but } A(\theta;x,y) = \frac{1}{2^{4/3}}\ .$$

(iii) If $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $\theta = (0,0)$ then

$$A(x;y,\theta) = \frac{1}{2}\,\frac{||y||}{(|y_1|^{3/2}+|y_2|^{3/2})^{2/3}}\,|x_1y_2-x_2y_1|$$

$$A(y;\theta,x) = \frac{1}{2}\,\frac{||x||}{(|x_1|^{3/2}+|x_2|^{3/2})^{2/3}}\,|x_1y_2-x_2y_1|$$

$$\text{and } A(\theta;x,y) = \frac{1}{2}\,\frac{||x-y||}{(|x_1-y_1|^{3/2}+|x_2-y_2|^{3/2})^{2/3}}\,|x_1y_2-x_2y_1|$$

Putting $A(x;y,\theta) = A(y;\theta,x)$, we get either $x_1y_2 = x_2y_1$ or $x_1y_1 = x_2y_2$. In the first case $A(x;y,\theta) = A(y;\theta,x) = A(\theta;y,x)=0$, and in the second case either $x = (a,0)$ and $y = (0,b)$ or $y_1 = k\ x_2$ and $y_2 = k\ x_1$, where k is some non-zero constant.

When $x = (a,0)$ and $y = (0,b)$ then

$A(x;y,\theta) = A(y;\theta,x) = A(\theta;x,y)$ implies either $a = 0$ or $b = 0$ i.e. the trivial case when $y_1 = k\ x_2$ and $y_2 = k\ x_1$, then again

$A(x;y,\theta) = A(y;\theta,x) = A(\theta;y,x)$ implies the trivial case.

3.6 <u>Area of type III</u>. We use the generalized inner product $<x,y>$ for $x,y \in X$, in the same way as inner product has been used in the first chapter to obtain the area of a triangle, and get the area of a triangle of type III in X as a sextuplet (since the generalized inner product is not commutative) of numbers, given by

$$\Delta_3(x,y,z) = \{C(x;y,z),\ C(x;z,y),\ C(y;z,x),\ C(y;x,z),\ C(z;x,y),\ C(z;y,x)\}$$

where $C(x;y,z) = \frac{1}{2}\ ||x-y||\ ||z-y||\ [1-<x\hat{-}y,z\hat{-}y>^2]^{1/2}$,

$$C(x;z,y) = \frac{1}{2}\ ||x-z||\ ||y-z||\ [1-<x\hat{-}z,y\hat{-}z>^2]^{1/2},$$

$$C(y;z,x) = \frac{1}{2}\ ||y-z||\ ||x-z||\ [1-<y\hat{-}z,x\hat{-}z>^2]^{1/2},$$

$$C(y;x,z) = \frac{1}{2}\ ||y-x||\ ||z-x||\ [1-<y\hat{-}x,z\hat{-}x>^2]^{1/2},$$

$$C(z;x,y) = \frac{1}{2}\ ||z-x||\ ||y-x||\ [1-<z\hat{-}x,y\hat{-}x>^2]^{1/2},$$

and $C(z;y,x) = \frac{1}{2}\ ||z-y||\ ||x-y||\ [1-<z\hat{-}y,x\hat{-}y>^2]^{1/2}$.

The following proposition is a simple outcome of the above definition.

<u>Proposition 3.6.1</u>. For $x,y,z \in X$,

(i) $C(x;y,z) = \frac{1}{2}\,||x-y||\;||z-y||\;[1-(J_{x\hat{-}y},z\hat{-}y)^2]^{1/2}$

(ii) $C(x;y,z) = C(x+w;y+w,z+w)$ for any $w \in X$.

(iii) $C(x;y,z)$ is a continuous function of x,y and z.

Proof. The proofs of (i) and (ii) are trivial. To prove (iii), we consider sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converging to x,y and z respectively.

$$C(x_n;y_n,z_n) = \frac{1}{2}\,[||x_n-y_n||^2||z_n-y_n||^2 - (J_{x_n-y_n},z_n-y_n)^2]^{1/2}$$

Putting $x_n-y_n = u_n$, $x-y = u$, $z_n-y_n = v_n$ and $z-y = v$,

$$\begin{aligned}|(J_{u_n},v_n)-(J_u,v)| &\le |(J_{u_n},v_n)-(J_{u_n},v)| + |(J_{u_n},v)-(J_u,v)| \\ &\le ||u_n||\;||v_n-v|| + |(J_{u_n}-J_u,v)|.\end{aligned}$$

Since J is norm to w^* continuous, the right hand side converges to zero as $n \to \infty$. Hence $(J_{u_n},v_n) \to (J_u,v)$ and then $C(x_n;y_n,z_n)$ converges to $C(x;y,z)$ as $n \to \infty$. This proves the continuity of C.

The following characterization of an inner product space also arises out of our definition of $\Delta_3(x,y,z)$.

Lemma 3.6.2. X is an inner product space if any two of the C's in the sextuplet $\Delta_3(x,y,z)$ are equal.

Remark. In view of the Proposition 3.6.1(ii), it is sufficient to prove that X is an inner product space when

(i) $C(x;\theta,y) = C(y;\theta,x)$, (ii) $C(x;\theta,y) = C(\theta;x,y)$,

(iii) $C(x;\theta,y) = C(\theta;y,x)$, (iv) $C(x;\theta,y) = C(y;x,\theta)$ and

(v) $C(x;\theta,y) = C(x;y,\theta)$ for all $x,y \in X$.

Proof (i) For any $x, y \in X$,

$$C(x;\theta,y) = C(y;\theta,x) \implies ||x||^2\,||y||^2 - (J_x,y)^2$$

$$= ||y||^2\,||x||^2 - (J_y,x)^2$$

$$\implies (J_x,y)^2 = (J_y,x)^2.$$

This implies orthogonality is symmetric. Hence if dimension $X \geq 3$ the proof is complete. To prove the result in general, let $||x|| = ||y|| = 1$ and $x \perp y$, we have

$$||x+y||^4 = (J_{x+y},x+y)^2 = \lfloor(J_{x+y},x) + (J_{x+y},y)\rfloor^2$$

$$= (J_{x+y},x)^2 + (J_{x+y},y)^2 + 2(J_{x+y}\cdot x)(J_{x+y},y)$$

$$= (J_x,x+y)^2 + (J_y,x+y)^2 \pm 2(J_x,x+y)(J_y,x+y)$$

$$= ||x||^4 + ||y||^4 \pm 2||x||^2\,||y||^2$$

$$= (||x||^2+||y||^2)^2 \text{ or } (||x||^2-||y||^2)^2.$$

But $||x+y||^4 = (||x||^2-||y||^2)^2 = 0$ is not possible, because $x = -y$ and $x \perp y$ imply $x = 0$. Hence

$$||x+y||^2 = ||x||^2+||y||^2.$$

This shows that $x \perp y$ implies $x \perp_p y$ for $||x|| = ||y|| = 1$. From Corollary 4.2.5 it follows that X is an inner product space.

(ii) $C(x;\theta,y) = C(\theta;x,y) \implies ||x||^2\,||y||^2 - (J_x,y)^2$

$$= ||x||^2\,||y-x||^2 - (J_x,y-x)^2.$$

Now $x \perp y \implies (J_x, y) = 0$

$$\implies \|x\|^2 \|y\|^2 = \|x\|^2 \|y-x\|^2 - \|x\|^4$$

$$\implies \|y-x\|^2 = \|x\|^2 + \|y\|^2$$

$$\implies x \perp_p y.$$

Hence X is an inner product space (Theorem 0.2.8(iv)).

(iii) $C(x;\theta,y) = C(\theta;y,x) \implies \|x\|^2 \|y\|^2 - (J_x,y)^2$

$$= \|y\|^2 \|x-y\|^2 - (J_y, x-y)^2.$$

Replacing y by -y,

$$\|x\|^2 \|y\|^2 - (J_x,y)^2 = \|y\|^2 \|x+y\|^2 - (J_y, x+y)^2.$$

Now $y \perp_i x \implies (J_y, x-y)^2 = (J_y, x+y)^2$

$$\implies (J_y, x) - \|y\|^2 = \pm\left[(J_y,x) + \|y\|^2\right]$$

$$\implies \text{either } y = 0 \text{ or } (J_y, x) = 0$$

$$\implies y \perp x \text{ since } y \neq 0.$$

Hence X is an inner product space (Theorem 0.2.8(iv))

(iv) $C(x;\theta,y) = C(y;x,\theta) \implies \|x\|^2 \|y\|^2 - (J_x,y)^2$

$$= \|y-x\|^2 \|x\|^2 - (J_{y-x}, x)^2$$

Now $x \perp_i y \implies (J_{y-x}, x)^2 = (J_{y+x}, x)^2$

$$\implies (J_u, v-u) = \pm (J_v, v-u), \text{ where } u = y-x \text{ and } v = y+x$$

$$\implies (J_u, v) - \|u\|^2 = \pm\left[\|v\|^2 - (J_v, u)\right]$$

$$\implies \text{either } 2\|u\|^2 = (J_u, v) + (J_v, u)$$

$$\text{or } (J_u, v) = (J_v, u).$$

But $(J_x - J_y, x-y) > 0$ for all $x \neq y$ [3], hence

$$2\|u\|^2 > (J_u, v) + (J_v, u).$$

Therefore, $x \perp_i y \Longrightarrow (J_{y-x}, y+x) = (J_{y+x}, y-x)$. But for any, $x, y \in X$, $\frac{\hat{x}+\hat{y}}{2} \perp_i \frac{\hat{x}-\hat{y}}{2}$, hence $(J_{\hat{y}}, \hat{x}) = (J_{\hat{x}}, \hat{y})$ and so $(J_y, x) = (J_x, y)$, giving that X is an inner product space.

(v) For $C(x;\theta,y) = C(x;y,\theta)$, a proof similar to that of (iv) can be constructed.

3.7 <u>Properties of the areas of three types</u>

Before taking up the study of triangle contractive (TC) maps in X, we make the following two lemmas which we would need while studying the fixtures of TC maps.

<u>Lemma 3.7.1.</u> If x and y are linearly independent vectors, $z \in X$ with $\|z\| \geq \varepsilon > 0$ then there exists $\xi > 0$ and $\eta > 0$ independent of such that

(i) $A(z;\theta,y) + A(z;x,\theta) \geq \xi$, and

(ii) $B(z;\theta,y) + B(z;x,\theta) \geq \eta$.

<u>Proof</u>. Let $A(z;\theta,y) = \frac{1}{2}\|y\|\,\|z-\lambda y\|$

and $A(z;x,\theta) = \frac{1}{2}\|x\|\,\|z-\mu x\|$

where λ and μ are such that $z-\lambda y \perp y$ and $z-\mu x \perp x$

$$A(z;\theta,y)+A(z;x,\theta) = \frac{1}{2}\|y\|\,\|z-\lambda y\| + \frac{1}{2}\|x\|\,\|z-\mu x\|$$

$$\geq \frac{1}{2}\min\{\|x\|,\|y\|\}\,\left[\|z-\lambda y\|+\|z-\mu x\|\right].$$

Clearly $\min\{||x||,||y||\} > 0$. We claim that $\operatorname{Inf}\{||z-\lambda y|| + ||z-\mu x||\} > 0$. If not, then there exists a sequence $\{z_n\}$ in X and sequences $\{\lambda_n\}$ and $\{\mu_n\}$ of numbers such that $z_n-\lambda_n y \perp y$, $z_n-\mu_n x \perp x$ and $||z_n-\lambda_n y|| + ||z_n-\mu_n x|| \to 0$. This implies that $z_n-\lambda_n y \to 0$ and $z_n-\mu_n x \to 0$. Hence $\lambda_n y-\mu_n x \to 0$, which is impossible in view of linear independence of x and y-unless $\lambda_n \to 0$ and $\mu_n \to 0$, but then $2\varepsilon = 0$ - a contradiction, and this establishes our claim, where

$$\xi = \frac{\operatorname{Inf}\{||z-\lambda y||+||z-\mu x|| : ||z|| \geq \varepsilon, z-\lambda y \perp y, z-\mu x \perp x\}}{\frac{1}{2}\min\{||x||,||y||\}} .$$

For (ii) also we can give a proof similar to that for (i).

<u>Lemma 3.7.2.</u> If X is a reflexive space, $x,y \in X$ are linearly independent and $z \in X$ with $||z|| \geq \varepsilon > 0$ then there exists $\eta > 0$ independent of z such that

$$C(x;\theta,z) + C(y;\theta,z) \geq \eta .$$

<u>Proof</u>. Suppose there is a sequence $\{z_n\}$ with $||z_n|| \geq \varepsilon$ and $C(x;\theta,z_n)+C(y;\theta,z_n) \to 0$ as $n \to \infty$, then $\frac{1}{2}||x|| \, ||z_n|| \, [1-(J_{\hat{x}},\hat{z}_n)^2]^{1/2} + \frac{1}{2}||y|| \, ||z_n|| \, [1-(J_{\hat{y}},\hat{z}_n)^2]^{1/2} \to 0$ which implies that $(J_{\hat{x}},\hat{z}_n)^2 \to 1$ and $(J_{\hat{y}},\hat{z}_n)^2 \to 1$. Since X is reflexive we may assume that $\{\hat{z}_n\}$ converges weakly to z, $||z|| \leq 1$. Hence $(J_{\hat{x}},z)^2 = 1 = (J_{\hat{y}},z)^2$, which gives $1 \leq ||\hat{x}|| \, ||z|| \leq 1$ and $1 \leq ||\hat{y}|| \, ||z|| \leq 1$, so that $||z|| = 1$. But in this case $\hat{x} = az$ and $\hat{y} = bz$ for some a and b. This is not possible since x and

y are linearly independent. Hence we may take

$\eta = \text{Inf}\ \{C(x;\theta,z) + C(y;\theta,z) : ||z|| \geq \varepsilon > 0\}$.

3.8 Triangle contractive maps and their continuity

Let $f : X \to X$, we define

(i) f is triangle expansion bounded of type I (TEB I) if for some $\alpha > 0$ and any $x,y,z \in X$

$||fx-fy|| \leq \alpha||x-y||, ||fy-fz|| \leq \alpha||y-z||$ and

$||fz-fx|| \leq \alpha||z-x||$

or $A(fx;fy,fz) \leq \alpha\, A(x;y,z)$, $A(fy;fz,fx) \leq \alpha\, A(y;z,x)$

and $A(fz;fx,fy) \leq \alpha\, A(z;x,y)$.

If $0 < \alpha < 1$, then we call f to be triangle contractive of type I (TC I).

(ii) f is triangle expansion bounded of type II (TEB II) and triangle contractive of type II (TC II) respectively if the conditions of (i) are satisfied with A replaced by B.

(iii) f is triangle expansion bounded of type III (TEB III) if for some $\alpha > 0$ and any $x,y,z \in X$, either $||fx-fy|| \leq \alpha\, ||x-y||$, $||fy-fz|| \leq \alpha\, ||y-z||$ and $||fz-fx|| \leq \alpha\, ||z-x||$

or $C(fx;fy,fz) \leq \alpha\, C(x;y,z)$, $C(fx;fz,fy) \leq \alpha\, C(x;z,y)$,

$C(fy;fz,fx) \leq \alpha\, C(y;z,x)$, $C(fy;fx,fz) \leq \alpha\, C(y;x,z)$,

$C(fz;fx,fy) \leq \alpha\, C(z;x,y)$ and $C(fz;fy,fx) \leq \alpha\, C(z;y,x)$.

If $0 < \alpha < 1$, then f is called triangle contractive of type III (TC III).

The following two theorems establish the existence of fixtures of TC maps in the absence of continuity. These results are also useful in proving the existence of fixtures in general. Now onwards we take $f : X \to X$.

Theorem 3.8.1. If f is TEB I or TEB II and is not continuous then fX is contained in a line.

Proof. Suppose f is TEB I and is not continuous at x, let $\{x_n\}$ be a sequence converging to x but $fx_n \not\to fx$, then there is an $\varepsilon > 0$ such that $\|fx_n - fx\| \geq \varepsilon$, $n = 1,2,\ldots$. We consider $L(fx_1, fx)$. If $fX \not\subset L(fx_1, fx)$, choose $z \in X$ such that $fz \notin L(fx_1, fx)$.

Since $\|fx_n - fx\| > \alpha\|x_n - x\|$ for sufficiently large n, we must have

$$(1)\quad \begin{aligned} A(fx_n; fx, fx_1) &\leq \alpha\, A(x_n; x, x_1) \\ \text{and } A(fx_n; fx, fz) &\leq \alpha\, A(x_n; x, z) \end{aligned}$$

By Proposition 3.2.1,

$$A(x_n; x, x_1) = A(x_n - x; \theta, x_1 - x) \to 0, \text{ and}$$

similarly $A(x_n; x, z) \to 0$.

On the other hand, since $fx_1 - fx$ and $fz - fx$ are linearly independent, by Lemma 3.7.1,

$$A(fx_n; fx, fx_1) + A(fx_n; fx, fz) \geq \xi > 0.$$

This leads to a contradiction, hence $L(fx_1, fx)$ must be a fixed line and $fX \subset L(fx_1, fx)$.

A similar proof works when f is TEB II.

Theorem 3.8.2. If f is TEB III and is not continuous then fX is contained in a line, provided X is reflexive.

Proof. We proceed on the same lines as in Theorem 3.8.1 except that we replace the inequalities (1) by

$$C(fx_1;fx,fx_n) \leq \alpha\, C(x_1;x,x_n)$$

$$\text{and} \quad C(fz;fx,fx_n) \leq \alpha\, C(z;x,x_n),$$

and then apply Proposition 3.6.1 and Lemma 3.7.2 to get the required result.

3.9 Fixtures of TC maps

Having seen in the first chapter that fixtures of TC maps exist in infinite-dimensional Hilbert spaces, of-course under certain conditions on the function, we do expect a similar result in the case of infinite-dimensional normed linear spaces and we prove the same under certain conditions on the function and space, and in one case we assume the space to be finite-dimensional. We begin by proving a result which is basic for the proof of our main results. We first give the following definition:

A map $f : X \to X$ is said to have the property (a) if "there exists a fixed point of f or a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ of numbers such that $fx_n = t_n x_n$, $t_n > 1$ for all n, $||x_n|| \to \infty$ and the sequence $\{\hat{x}_n\}$ converges".

A continuous TC map on a finite-dimensional space clearly has the property (a). Apart from this, we have shown

in Theorem 3.9.5 that there are infinite-dimensional situations also in which TC maps have the property (a).

<u>Theorem 3.9.1</u>. Let X be a Banach space, f is a TC map without a fixed point on X with the property (a), then there exists a direction such that lines parallel to this direction are mapped on lines parallel to it.

<u>Proof</u>. The proofs differ for each type of TC map.

(i) Let f be TC I. For sufficiently large n and for any $x \varepsilon X$.

(1) $$||fx_n - fx|| > \alpha \, ||x_n - x||,$$

hence $A(fy; fx_n, fx) \leq \alpha \, A(y; x_n, x)$ for any $y \varepsilon X$, i.e.

$$||fx_n - fx|| \; ||fy - fx + \lambda_n(fx - fx_n)||$$

$$\leq \alpha \, ||x_n - x|| \; ||y - x + \lambda_n'(x - x_n)||$$

Since $\dfrac{||x_n - x||}{||fx_n - fx||} \leq 1/\beta$ for sufficiently large n,

(2) $$||fy - fx + \lambda_n(fx - fx_n|| \leq \frac{\alpha}{\beta} \, ||y - x + \lambda_n'(x - x_n)||.$$

Here $fy - fx + \lambda_n(fx - fx_n) \perp fx - fx_n$, hence

$$||fy - fx + \lambda_n(fx - fx_n)|| \leq ||fy - fx||,$$

and thus the sequence $\{||fy - fx + \lambda_n(||fx - fx_n||)(fx \hat{-} fx_n)||\}$ is bounded; it either converges or has a convergent subsequence. By hypothesis $\{\hat{x}_n\}$ converges to some w and hence $\{x_n \hat{-} x\}$ and $\{fx_n \hat{-} fx\}$ also converge to w. Taking limit as $n \to \infty$, the inequality (2) gives

(3) $$||fy - fx + \lambda w|| \leq ||y - x + \lambda' w||,$$

where $fy - fx + \lambda w \perp w$ and $y - x + \lambda' w \perp w$.

From this inequality we gain conclude that lines parallel to w are mapped on lines parallel to w.

<u>Theorem 3.9.2.</u> Let X be any Banach space in which orthogonality is left additive and f is TC I on X and has the property (a), then f has a fixture.

<u>Proof</u>. Following the arguments of the proof of Theorem 3.9.1(i) we get the inequality (3) as

$$||fy-fx-\mu w|| \leq \alpha \, ||y-x-\mu' w|| \text{ for all } x,y \in X.$$

Here $fy-fx-\mu w \perp w$ and $y-x-\mu' w \perp w$. The latter relation implies that $||y-x-\mu' w|| \leq ||y-x||$, hence

(1) $$||fy-fx-\mu w|| \leq \alpha \, ||y-x-\mu' w|| \leq \alpha \, ||y-x||.$$

We define $T : X \to X$ where for $x \in X$, $Tx = fx-\mu_1 w$, μ_1 is chosen so that $fx-\mu_1 w \perp w$.
Also for any $y \in X$, $Ty = fy-\mu_2 w$ where $fy-\mu_2 w \perp w$. By left additivity of the orthogonality, $fx-fy-(\mu_1-\mu_2)w \perp w$, and using left uniqueness of the orthogonality, $\mu_1-\mu_2 = -\mu$, hence

$$||Tx-Ty|| = ||fx-fy-\mu w|| \leq \alpha \, ||y-x||.$$

Therefore, T is a contraction and has a fixed point $x_o \in X$ so that $Tx_o = fx_o-\mu_o w = x_o$.

Now if L is the line through x_o and fx_o then it is parallel to w and it is mapped on itself under f, by Theorem 3.9.1. Thus L is fixed under f.

Theorem 3.9.3. If X is a Banach space and f is TC II on X, with the property (a), then f has a fixture.

Proof. Proceeding as in the proof of the Theorem 3.9.1(ii) we get the inequality (4′) which we re-write as

(1) $||fx-fy-(J_w, fx-fy)w|| \leq \alpha\, ||x-y-(J_w, x-y)w||.$

Let $M = \{z \in X : w \perp z\} = \{z \in X : (J_w, z) = 0\}$, Then M is a closed hyperplane in X. We define $P_M : X \to M$, where for $x \in X$, $P_M x = x-(J_w, x)w$ clearly $P_M x \in M$, we call P_M the projection On M. From the inequality (1), for any $x, y \in M$, we have

$$||P_M fx-P_M fy|| \leq \alpha\, ||P_M x-P_M y|| = \alpha||x-y||.$$

This shows that $P_M f$ is a contraction on M. Since M is closed subspace of X, $P_M f$ has a fixed point $x_o \in M$ so that $P_M fx_o = x_o$.

Let L be a line through x_o and parallel to w, then applying (1) to x_o and any $u \in L$,

$$\begin{aligned} ||P_M fu-P_M fx_o|| &\leq \alpha||P_M u-x_o|| = \alpha||u-(J_w, u)w-x_o|| \\ &= \alpha||u-x_o|| \;\; ||u\hat{-}x_o-(J_w, u\hat{-}x_o)w|| \\ &= 0. \end{aligned}$$

$$\begin{aligned} \text{Now } P_M fu-P_M fx_o = 0 &\Longrightarrow fu-(J_w, fu)w = x_o \\ &\Longrightarrow fu-x_o = (J_w, fu)w \\ &\Longrightarrow fu \in L. \end{aligned}$$

Hence L is fixed under f.

<u>Remark</u>. To arrive at the desired conclusion we need the left additivity of orthogonality in Theorem 3.9.2. In a two-dimensional strictly convex space X left additivity of the orthogonality can be shown as follows :

For $\theta \neq x$ and $\theta \neq y$, let $\theta \neq z \in x$ such that $x \perp z$ and $y \perp z$, then x and z are linearly independent, suppose $y = \lambda x + \mu z$ for some λ and μ. If $\lambda = 0$ then obviously $\mu = 0$ - not possible, so $\lambda \neq 0$, dividing by λ we have $x + \frac{\mu}{\lambda} z \perp z$. Now by strict convexity $\mu = 0$, hence $y = \lambda x$ and then $x+y = (1+\lambda)x \perp z$.

However, in spaces of higher dimension the condition appears to be too strong, possibly satisfied only in inner product spaces. In the following theorem we replace this condition by a condition on f and prove the existence of fixture when f is either TC I or TC III on a finite-dimensional space.

<u>Theorem 3.9.4</u>. Let X be a finite-dimensional Banach space, f is either TC I or TC III for which the following holds :

'There exists a bounded sequence $\{z_n\}$ in X such that $||fz_n - z_n|| \to 0$ as $n \to \infty$'.

Then f has a fixture.

<u>Proof</u>. In view of Theorems 3.8.1 and 3.8.2 we can assume f to be continuous. Since X is finite-dimensional f has property (a).

(i) Let f be TC I. Proceeding as in Theorem 3.9.2 we get the inequality (1) of that theorem as

$$||fy - fx - \mu w|| \leq \alpha\, ||y - x - \mu' w|| \leq \alpha ||y - x|| \text{ for any } x, y \in X.$$

Putting $y = fz_n$ and $x = z_n$,

$$||f^2z_n-fz_n-\lambda_n w|| \leq \alpha\ ||fz_n-z_n-\lambda'_n w|| \leq \alpha||fz_n-z_n||.$$

Since $\{z_n\}$ is a bounded sequence, it either converges or has a convergent subsequence converging to z_o. Taking limit as $n \to \infty$

(1) $\quad ||f^2z_o-fz_o-\lambda w|| \leq \alpha\ ||fz_o-z_o-\lambda' w|| \leq 0$

where λ and λ' are limits of the bounded sequences $\{\lambda_n\}$ and $\{\lambda'_n\}$ (or their subsequences) respectively.

The inequality (1) implies that the line L through z_o and fz_o is parallel to w and is fixed.

(ii) Let f be TC III. Following the lines of proof of Theorem 3.9.1(iii) we get the inequality (5′) of that theorem as

$$||fy-fx||^2-(J_w,fy-fx)^2 \leq \alpha^2\ [\ ||y-x||^2-(J_w,y-x)^2\ ]$$

for any $x,y \in X$.

Putting $y = fz_n$ and $x = z_n$, and taking limit

$$||f^2z_o-fz_o||^2 - (J_w,f^2z_o-fz_o)^2 \leq 0$$

where z_o is as in (i) above.

Now if L is a line through fz_o parallel to w, then $f^2z_o \in L$, and hence it follows from Theorem 3.9.1(iii) that L is fixed under f.

Theorem 3.9.5. In a reflexive Banach space X,

(i) If f is TC I or TC III, completely continuous (CC) and is such that $f\theta \neq fx$ for any $||x|| \leq 1$, then f has the property (a).

(ii) If f is TC II, CC and is such that $f\theta \neq fx$ for any $||x|| \leq 1$, then f has the property (a) provided $\alpha < \frac{1}{2}$ and J is weak to weak* continuous.

Proof. If f has no fixed point then applying Schauder's fixed point theorem to the composite function $T_n f$, where T_n is the radial retraction on the sphere $B_n(\theta)$ we obtain a sequence $\{x_n\}$ such that $fx_n = \lambda_n x_n$ with $\lambda_n > 1$ and $||x_n|| \to \infty$. Putting $\hat{x}_n = \frac{x_n}{||x_n||}$ without loss of generality we assume that $\hat{x}_n \rightharpoonup w$. We will first show that $w \neq \theta$:

Case I. f is TC I, then by inequality (2) of Theorem 3.9.1 we have for any $x,y \in X$, $||fy-fx+\lambda_n(fx-fx_n)|| \leq \alpha||y-x+\lambda_n'(x-x_n)|| \leq \alpha||y-x||$. Therefore by passing on to a subsequence of $\{fy-fx+\lambda_n(fx-fx_n)\}$ and using the weak lower semi-continuity of the norm we get for some λ

$$||fy-fx+\lambda w|| \leq \alpha||y-x||.$$

If $w = \theta$, f becomes a contraction, which is false (f has no fixed point).

When f is TC III, we use (5) in place of (2) to prove $w \neq \theta$.

Case II. If f is TC II then by inequality (4) of Theorem 3.9.1 we have

$$||fx-fy||\ ||fx\hat{-}fy-(J_{fx_n\hat{-}fx}, fx\hat{-}fy)(fx_n\hat{-}fx)||$$

$$\leq \alpha||x-y||\ ||x\hat{-}y-(J_{x_n\hat{-}x}, x\hat{-}y)(x_n\hat{-}x)|| \leq 2\alpha||x-y||$$

Weak to weak* continuity of J together with weak lower semi-continuity of norm again yields

$$||fx-fy||\ ||fx\hat{-}fy-(J_w, fx\hat{-}fy)w||$$

$$\leq 2\alpha||x-y||.$$

Hence as above $w \neq \theta$, since $\alpha < \frac{1}{2}$.

Since f is TC of any of the three types, the inequality (1) of Theorem 3.9.1 implies that the collinear points θ, $\hat{x}_n$ and x_n are mapped on collinear points.

Therefore, $f\hat{x}_n = \alpha_n\ fx_n + (1-\alpha_n)\ f\theta$, for some α_n's

$$= \alpha_n \lambda_n x_n + (1-\alpha_n)\ f\theta$$

$$= \alpha_n \lambda_n ||x_n||\ \hat{x}_n + (1-\alpha_n)\ f\theta.$$

Since f is CC we may assume $f\hat{x}_n \to fw$, hence α_n's can not be unbounded. Also since $\lambda_n||x_n|| \to \infty$, we can assume $\alpha_n \to 0$, and then

$$fw = y+f\theta, \text{ where } y = \lim_{n\to\infty} \alpha_n\lambda_n ||x_n||\ \hat{x}_n.$$

Here $y \neq \theta$ by hypothesis, hence $y = tw$ for some t, thus $\hat{x}_n \to w$, and f has the property (a).

3.10 Existence of fixtures under different conditions

Theorem 3.10.1. If f is TC I or TC II and L is a line with $x,y \in L$ such that $fx, fy \in L$ and $||fx-fy|| \geq \beta||x-y|| > 0$ where

$\beta > \alpha$, then L is a fixed line. Further $f^n w \to L$ for any $w \in X$.

<u>Proof</u>. Let f be TC I. For any $w \in L$, we have by hypothesis,

$$A(fw;fx,fy) \leq \alpha\, A(w;x,y) = 0$$

hence $fw \in L$ and it is fixed.

Again for any $w \in X$,

$A(f^{n+1}w;fx,fy) \leq \alpha\, A(f^n w;x,y)$ implies that

$$||fx-fy||\ ||f^{n+1}w - (\lambda_{n+1}fx+(1-\lambda_{n+1})fy)||$$

$$\leq \alpha||x-y||\ ||f^n w-(\lambda_n fx+(1-\lambda_n)fy)||, \text{ since } L(x,y) = L(fx,fy).$$

Also since $||fx-fy|| \geq \beta||x-y||$

$$||f^{n+1}w-(\lambda_{n+1}fx+(1-\lambda_{n+1})fy)|| \leq \frac{\alpha}{\beta}\ ||f^n w-(\lambda_n fx+(1-\lambda_n)fy)||$$

Iterating n times,

$$||f^{n+1}w-(\lambda_{n+1}fx+(1-\lambda_{n+1})fy)|| \leq (\frac{\alpha}{\beta})^n\ ||fw-(\lambda_1\ fx+(1-\lambda_1)fy)||.$$

Since $fx \neq fy$ and $fw-(\lambda_1 fx+(1-\lambda_1)fy) \perp fx-fy$, hence

$$||fw-(\lambda_1 fx+(1-\lambda_1)fy)|| \leq ||fw-fy||.$$

Also $(\frac{\alpha}{\beta})^n \to 0$ as $n \to \infty$. Taking limit, it follows that

$f^n w \to$ ~~$\lambda fx+(1-\lambda)fy$ as~~ L, ~~where λ is the limit of the bounded sequence $\{\lambda_n\}$ (or its subsequence)~~.

A similar proof can be given when f is TC II.

<u>Remark</u>. The first part of Theorem 3.10.1 trivially holds when f is TC III, but the second part can be proved when x and y are

fixed points, as has been done in Proposition 3.11.1.

<u>Theorem 3.10.2</u>. If f is a TC map of any of the three types, x, y are distinct points of X with $fx = y$ and $fy = x$ then $L(x,y)$ is a fixed line.

<u>Proof</u>. If $L(x,y)$ is not fixed we can get $z \in L(x,y)$ where $fz \notin L(x,y)$ and since $||fx-fy|| > \alpha||x-y||$, we must have, assuming f to be TC I,

$$A(fz;fx,fy) \leq \alpha\, A(z;x,y).$$

But $A(z;x,y) = 0$ and $A(fz;fx,fy) \neq 0$ gives a contradiction. Similarly we can prove when f is TC II or TC III.

<u>Theorem 3.10.3</u>. Let f be a TEB map of any of the three types and $w \in X$ has the property that every neighbourhood of w contains a point x and fx, then either w is a fixed point or $L(w,fw)$ is a fixed line containing fX.

<u>Proof</u>. Take f to be TEB I and assume that neither w is a fixed point nor $L(w,fw)$ is a fixed line, choose $z \in X$ where $fz \notin L(w,fw)$.

By hypothesis we can get a sequence $\{x_n\}$ such that $x_n \to w$ and $fx_n \to w$. Now for sufficiently large n, $||fx_n-fw|| > \alpha||x_n-w||$, hence $A(fx_n;fw,fz) \leq \alpha\, A(x_n;w,z)$, where z is as chosen above. This leads to a contradiction since $A(x_n;w,z) \to 0$ but $A(fx_n;fw,fz) \geq \xi > 0$ for all n.

Similar proofs can be given when f is TEB II or TEB III.

3.11 Consequences of the existence of more than one fixture

Proposition 3.11.1. Let f be a TC map of any of the three types with x and y as its distinct fixed points, then $L = L(x,y)$ is a fixed line. Further $f^n w \to L$ for any $w \in X$.

Proof. If f is TC I or TC II then the result follows from Theorem 3.10.1.

Let f be TC III. For any $z \in L$, we have by hypothesis

$$C(fz;fx,fy) \leq \alpha\, C(z;x,y)$$

i.e. $\|fz-x\|[1-(J_{fz\hat{-}x}, x\hat{-}y)^2]^{1/2} \leq \alpha\|z-x\|[1-(J_{z\hat{-}x}, x\hat{-}y)^2]^{1/2}$.

This gives either $fz = x$ or $(J_{fz\hat{-}x}, x\hat{-}y)^2 = 1$, in either case $fz \in L$.

To prove the second part take any $w \in X$ and replace fz by $f^{n+1}w$ in the above inequality, then

$$\|f^{n+1}w-x\|[1-(J_{f^{n+1}w\hat{-}x}, x\hat{-}y)^2]^{1/2} \leq \alpha\|f^n w-x\|[1-J_{f^n w\hat{-}x}, x\hat{-}y)^2]^{1/2}.$$

By induction,

$$\|f^{n+1}w-x\|[1-(J_{f^{n+1}w\hat{-}x}, x\hat{-}y)^2]^{1/2} \leq \alpha^n\|fw-x\|[1-(J_{fw\hat{-}x}, x\hat{-}y)^2]^{1/2}.$$

Since $\alpha < 1$, $\alpha^n \to 0$ as $n \to \infty$, hence $f^n w \to L$.

Proposition 3.11.2. Let f be a TC map of any of the three types, and x,y,z are its fixed points, then they are collinear.

Proof. Let f be TC I (similar proof works when f is TC II or TC III). Since $\|fx-fy\| > \alpha\,\|x-y\|$,

$$A(fz;fx,fy) = A(z;x,y) \leq \alpha\, A(z;x,y).$$

Thus $A(z;x,y) = 0$, hence x,y,z are collinear.

<u>Proposition 3.11.3.</u> If f is a TC map of any of the three types then limits of convergent sequences of iterates are collinear whenever such sequences exist.

<u>Proof</u>. Let f be TC I, $\{f^n w_1\}$, $\{f^n w_2\}$ and $\{f^n w_3\}$ be three sequences of iterates converging to v_1, v_2 and v_3 respectively. Assuming v_1, v_2, v_3 to be distinct,

$$||f^n w_i - f^n w_j|| \to ||v_i - v_j|| > 0 \text{ for all } i \text{ and } j,$$

hence for sufficiently large n,

$$(1) \qquad ||f^n w_i - f^n w_j|| > \alpha\, ||f^{n-1} w_i - f^{n-1} w_j||,\ i,j = 1,2,3.$$

Also $A(f^n w_1; f^n w_2, f^n w_3) = \frac{1}{2}\, ||f^n w_2 - f^n w_3||\ ||f^n w_1 - f^n w_2 + {}$

$$+ \lambda_n\, (f^n w_2 - f^n w_3)||$$

where $f^n w_1 - f^n w_2 + \lambda_n (f^n w_2 - f^n w_3) \perp f^n w_2 - f^n w_3$, and so $\{\lambda_n\}$ is bounded. Taking limit as $n \to \infty$,

$$A(f^n w_1; f^n w_2, f^n w_3) \to A(v_1; v_2, v_3).$$

If $A(v_1; v_2, v_3) > 0$, then for sufficiently large n,

$$(2) \qquad A(f^n w_1; f^n w_2, f^n w_3) > \alpha\, A(f^{n-1} w_1; f^{n-1} w_2, f^{n-1} w_3).$$

The inequalities (1) and (2) together lead to a contradiction. Hence $A(v_1; v_2, v_3) = 0$ and so v_1, v_2, v_3 are collinear.

A similar proof can be given when f is TC II.

When f is TC III, proceeding as above we can get the inequality (1). Also

$$C(f^n w_1; f^n w_2, f^n w_3)$$

$$= \frac{1}{2}\left[||f^n w_1 - f^n w_2||^2 ||f^n w_3 - f^n w_2||^2 - (J_{f^n w_1 - f^n w_2}, f^n w_3 - f^n w_2)^2\right]^{\frac{1}{2}}.$$

Since J is norm to w^* continuous, taking limit as $n \to \infty$

$$C(f^n w_1; f^n w_2, f^n w_3) \to C(v_1; v_2, v_3).$$

This again gives an inequality similar to (2), hence v_1, v_2, v_3 are collinear.

Proposition 3.11.4. If f is TC I or TC II, L_1 and L_2 are two distinct lines satisfying the conditions of Theorem 3.10.1, then f has one and only one fixed point p - the intersection of L_1 and L_2. Further $f^n w \to p$ for each $w \in X$.

Proof. By Theorem 3.10.1 both L_1 and L_2 are fixed lines and $f^n w \to L_1$ and $f^n w \to L_2$ for each $w \in X$. Hence L_1 and L_2 intersect in p which is a fixed point and $f^n w \to p$ for each $w \in X$, and this implies that if q is any other fixed point then $q = p$.

Proposition 3.11.5. If f is TC I or TC II on a Banach space X, L_1 and L_2 are distinct fixed lines of f then there are fixed points x_1 and x_2 on L_1 and L_2 respectively, x_1 and x_2 may be coincident.

Proof. If f is contractive on L_1 as well as on L_2 then there are fixed points x_1 and x_2 on L_1 and L_2 respectively.

Suppose f is not contractive on L_1 then it satisfies the conditions of Theorem 3.10.1 on L_1, hence $f^n w \to L_1$ for every $w \in X$, in particular for each $w \in L_2$. Since L_2 is fixed, L_1 and L_2 intersect and the point of intersection is a fixed point.

<u>Proposition 3.11.6</u>. If f is TC I or TC II on a Banach space X, with a fixed line L and a fixed point p not on L, then f is contractive on L and so has a fixed point on L.

<u>Proof</u>. If f is not contractive on L, then by Theorem 3.10.1, $f^n w \to L$ for each $w \in X$, hence $p \in L$ - a contradiction.

From the above results the following theorem can be proved easily :

<u>Theorem 3.11.7</u>. Let f be TC I or TC II.

(i) The point of intersection (if any) of two different fixed lines is a fixed point.

(ii) If f has no fixed point then it has atmost one fixed line, and if it does have one such line L, then $f^n w \to L$ for each $w \in X$.

(iii) If f has exactly one fixed point p then its fixed lines (if any) all pass through p.

(iv) If f has two or more fixed points then they will all lie on a fixed line L. Moreover any other fixed line M will intersect L and $f^n w \to L$ for each $w \in X$.

<u>Proof</u> (i) Trivial.

(ii) If f has a fixed line L then f can not be contractive on L, otherwise it will have a fixed point also. By Theorem 3.10.1, $f^n w \to L$ for any $w \in X$.

Moreover if M is any other fixed line then by Proposition 3.11.5 it will intersect L in a fixed point - not possible.

(iii) If L is a fixed line not passing through p, then by Proposition 3.11.6 there is another fixed point on L, which contradicts the hypothesis.

(iv) By proposition 3.11.2 all fixed points lie on a line L, which is fixed by Theorem 3.10.1 and $f^n w \to L$ for each $w \in X$. Moreover any other fixed line M will intersect L by Proposition 3.11.5.

3.12 Examples of TC maps

We have already proved that if in a two-dimensional space orthogonality is symmetric then $\Delta_1(x,y,z) = \Delta_2(x,y,z) = \Delta(x,y,z)$ (say) and $\Delta(x,y,z) = \frac{1}{2} \,|(x_1y_2-x_2y_1)+(y_1z_2-y_2z_1)+(z_1x_2-z_2x_1)|$ where $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $z = (z_1,z_2)$ with respect to two orthogonal unit vectors.

This expression for $\Delta(x,y,z)$ is the same as that for a two-dimensional Hilbert space and hence all the results and examples concerning fixtures of TC and TEB maps for Hilbert spaces hold here as well.

As a concrete example we consider, $X = \ell_3^2$, where for $x = (x_1,x_2) \in X$, we define

(ii) If f has a fixed line L then f can not be contractive on L, otherwise it will have a fixed point also. By Theorem 3.10.1, $f^n w \to L$ for any $w \in X$.

Moreover if M is any other fixed line then by proposition 3.11.5 it will intersect L in a fixed point - not possible.

(iii) If L is a fixed line not passing through p, then by Proposition 3.11.6 there is another fixed point on L, which contradicts the hypothesis.

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This expression for $\Delta(x,y,z)$ is the same as that for a two-dimensional Hilbert space and hence all the results and examples concerning fixtures of TC and TEB maps for Hilbert spaces hold here as well.

As a concrete example we consider, $X = \ell_3^2$, where for $x = (x_1,x_2) \in X$, we define

$$||x|| = \begin{cases} (|x_1|^3+|x_2|^3)^{1/3} & \text{if } x_1 > 0,\ x_2 > 0 \text{ or } x_1 < 0,\ x_2 < 0 \\ (|x_1|^{3/2}+|x_2|^{3/2})^{2/3} & \text{if } x_1 > 0, x_2 < 0 \text{ or } x_1 < 0, x_2 > 0. \end{cases}$$

With this norm the orthogonality is symmetric in X and the space is smooth and strictly convex. We define

$f : X \to X$ where $f(x_1,x_2) = (x_1, \frac{x_2}{3})$ for any $(x_1,x_2) \in X$, then for any $x,y,z \in X$,

$$||fx-fy|| \leq ||x-y||, ||fy-fz|| \leq ||y-z||, ||fz-fx|| \leq ||z-x||,$$

and $\Delta(fx,fy,fz) = \frac{1}{3}\,\Delta(x,y,z)$

Hence f is TC.

We next take the example of a strictly convex and smooth space in which the orthogonality is not symmetric. $X = \ell_p^2, p > 2$, with usual norm is such a space. We first derive an expression for $A(z;x,y)$ in this space where we know that

$$(J_x,y) = \frac{|x_1|^{p-1}\ y_1 \ \text{sgn}\ x_1 + |x_2|^{p-1}\ y_2\ \text{sgn}\ x_2}{||x||^{p-2}}.$$

$A(y;\theta,x) = \frac{1}{2}\ ||x||\ ||y-\lambda x||$, where $(J_{y-\lambda x},x) = 0$, which means that

$$|y_1-\lambda x_1|^{p-1}\ x_1\ \text{sgn}\ (y_1-\lambda x_1)+|y_2-\lambda x_2|^{p-1}\ x_2\ \text{sgn}\ (y_2-\lambda x_2) = 0.$$

Only two possibilities are there :

(i) x_1, x_2 are of same sign and $(y_1-\lambda x_1)$, $(y_2-\lambda x_2)$ are of opposite signs. We can assume, without loss of generality, that $x_1 > 0$ and $x_2 > 0$.

(ii) x_1, x_2 have opposite signs and $(y_1-\lambda x_1)$, $(y_2-\lambda x_2)$ have same sign.

When (i) is there we can write,

$$(y_1-\lambda x_1)\, x_1^{\frac{1}{p-1}} = -(y_2-\lambda x_2)\, x_2^{\frac{1}{p-1}}, \text{ which gives}$$

$$\lambda = \frac{y_1\, x_1^{\frac{1}{p-1}} + y_2 x_2^{\frac{1}{p-1}}}{x_1^{\frac{p}{p-1}} + x_2^{\frac{p}{p-1}}}, \text{ and then}$$

$$y_1-\lambda x_1 = \frac{x_2^{\frac{1}{p-1}}(y_1x_2-y_2x_1)}{x_1^{\frac{p}{p-1}} + x_2^{\frac{p}{p-1}}} \text{ and } y_2-\lambda x_2 = \frac{x_1^{\frac{1}{p-1}}(y_2x_1-y_1x_2)}{x_1^{\frac{p}{p-1}} + x_2^{\frac{p}{p-1}}},$$

$$\text{hence } A(y;\theta,x) = \frac{1}{2}\, \frac{(x_1^p+x_2^p)^{\frac{1}{p}}\left(x_1^{\frac{p}{p-1}} + x_2^{\frac{p}{p-1}}\right)^{\frac{1}{p}}}{x_1^{\frac{p}{p-1}} + x_2^{\frac{p}{p-1}}}\, |x_1y_2-x_2y_1|$$

$$= \frac{1}{2}\, \frac{||x||_p}{||x||_q}\, |x_1y_2-x_2y_1|, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We get the same expression for $A(y;\theta,x)$ when (ii) is there.

It can be easily seen that

$$(1) \quad \frac{1}{2^{\frac{1}{q}-\frac{1}{p}}} \leq \frac{(|x_1|^p + |x_2|^p)^{\frac{1}{p}}}{(|x_1|^q + |x_2|^q)^{\frac{1}{q}}} \leq 1, \text{ for } p > 2 \text{ (and so } q < p)$$

$$\text{Now, } A(z;x,y) = \frac{1}{2}\, \frac{||y-x||_p}{||y-x||_q}\, |(x_1y_2-x_2y_1)+(y_1z_2-y_2z_1)+(z_1x_2-z_2x_1)|.$$

Similar expressions for $A(x;y,z)$ and $A(y;z,x)$ can also be obtained.

We define $f : \ell_p^2 \to \ell_p^2$, $f(x_1, x_2) = (x_1+1, \frac{x_2}{2})$ for any $(x_1, x_2) \in \ell_p^2$. Then for any $x, y, z \in \ell_p^2$,

$$||fx-fy|| \leq ||x-y||, ||fy-fz|| \leq ||y-z||, \ ||fz-fx|| \leq ||z-x||,$$

and
$$A(fz;fx,fy) = \frac{1}{2}\,\frac{||fy-fx||_p}{||fy-fx||_q}\,\frac{1}{2}\,|(x_1y_2-x_2y_1)+(y_1z_2-y_2z_1) + (z_1x_2-z_2x_1)|$$

$$\leq \frac{1}{2^2}\,|(x_1y_2-x_2y_1)+(y_1z_2-y_2z_1)+(z_1x_2-z_2x_1)|,$$

(by inequality (1))

$$\leq \frac{1}{2^2}\,\frac{\frac{||y-x||_p}{||y-x||_q}}{\frac{1}{2^{\frac{1}{q}-\frac{1}{p}}}}\,|(x_1y_2-x_2y_1)+(y_1z_2-y_2z_1)+ + (z_1x_2-z_2x_1)|$$

(again by inequality (1))

$$\leq \frac{1}{2^{1-\frac{1}{q}+\frac{1}{p}}}\ A(z;x,y)$$

$$\leq \frac{1}{2^{\frac{2}{p}}}\,A(z;x,y).$$

Similarly, $A(fx;fy,fz) \leq \frac{1}{2^{\frac{2}{p}}}\,A(x;y,z)$

and $A(fy;fz,fz) \leq \frac{1}{2^{\frac{2}{p}}}\,A(y;z,x)$.

Therefore f is TC I and has x_1-axis as the only fixture.

In the same space we can also look for an expression for $B(z;x,y)$. As above we start with

$$B(y;\theta,x) = \frac{1}{2}\ ||x||\ ||y - \frac{(J_x,y)}{||x||^2}\ x||$$

$$= \frac{1}{2}\ ||x||\ ||(y_1,y_2) - \frac{|x_1|^{p-1}\ y_1\ \mathrm{sgn}\ x_1 + |x_2|^{p-1} y_2 \mathrm{sgn}\ x_2}{||x||^p}$$

$$x\ (x_1,x_2)||$$

$$= \frac{1}{2||x||^{p-1}}\ ||(y_1||x||^p - |x_1|^p y_1 - |x_2|^{p-1} y_2 x_1\ \mathrm{sgn}\ x_2,$$

$$y_2||x||^p - |x_2|^p y_2 - |x_1|^{p-1} y_1 x_2\ \mathrm{sgn}\ x_1)||$$

$$= \frac{1}{2||x||^{p-1}}\ [|y_1 x_2^p - x_2^{p-1}\ y_2 x_1|^p + |y_1 x_1^p - x_1^{p-1}\ y_1 x_2|^p]^{\frac{1}{p}}$$

(we have considered the case when $x_1 > 0,\ x_2 > 0$, similarly other cases can be dealt with).

$$= \frac{1}{2}\ \frac{[x_1^{p(p-1)} + x_2^{p(p-1)}]^{\frac{1}{p}}}{[x_1^p + x_2^p]^{\frac{p-1}{p}}}\ |x_1 y_2 - x_2 y_1|$$

$$= \frac{1}{2}\ \frac{||x||^{p-1}_{p^2/q}}{||x||^{p-1}_p}\ |x_1 y_2 - x_2 y_1|.$$

Again it is easy to verify that

$$(2) \qquad \frac{1}{2^{\frac{1}{q} - \frac{1}{p}}} \leq \frac{||x||^{p-1}_{p^2/q}}{||x||^{p-1}_p} \leq 1,\ \text{for}\ p > 2.$$

$$\text{Now}\ B(z;x,y) = \frac{1}{2}\ \frac{||y-x||^{p-1}_{p^2/q}}{||y-x||^{p-1}_p}\ |(x_1 y_2 - x_2 y_1) + (y_1 z_2 - y_2 z_1)$$

$$+ (z_1 x_2 - z_2 x_1)|,$$

and two similar expressions for $B(x;y,z)$ and $B(y;z,x)$.

In view of the inequality (2) the mapping $f(x_1,x_2) = (x_1+1, \frac{x_2}{2})$ is TC II also, and f has a fixture as already seen.

We also note that some more examples of functions as given in [10] trivially turn out to be TC I and TC II on ℓ_p^2 $(p > 2)$.

CHAPTER IV

SOME GEOMETRIC CHARACTERIZATIONS OF INNER PRODUCT SPACES

4.1 Introduction. In this chapter our object of study is geometric characterizations of inner product spaces amongst normed linear spaces. Some such characterizations arose out of our consideration of various definitions of the area of a triangle in the third chapter. Here we would be concerned with refinements of some of the known characterizations. Day's [5] refinement 'rhombi suffice' of the parallelogram law is among the well-known ones. There are some characterizations which have been described in terms of orthogonality. For example, it is known that if in a normed linear space X orthogonality implies isosceles orthogonality then X must be an inner product space. We improve upon this characterization in the same sense as Day did for the parallelogram law, and in the same vein some characterizations due to Day [6], Kapoor and Prasad [22] and Holub [18] have been refined.

4.2 The main result and its corollaries

Theorem 4.2.1. For a real normed linear space X the following are equivalent :

(i) X is an inner product space.

(ii) $x, y \in S$, $x \perp y \implies x \perp_i y$

(iii) $m(X) = \sqrt{2}$ (see Definition 0.2.13).

Proof (i) ==> (ii) is obvious and (iii) ==> (i) is proved in [11]. We have to prove (ii) ==> (iii). This proof consists of many steps which we give below as lemmas.

Lemma 4.2.2. If X satisfies (ii) then it is strictly convex.

Proof. Assuming X is not strictly convex, choose [19, Theorem 4.3] $x, y \in S$ such that $x \perp y$ and $\alpha y+x \perp y$ where $\alpha > 0$ is chosen to be the largest such number. Let $\phi(t) = ||x+ty||$, we see that ϕ is a convex function of t, and since

$$1 = ||x|| \leq ||x+\alpha y|| \leq ||x+\alpha y+\lambda y|| \text{ for } \lambda \in R.$$

Putting $\lambda = -\alpha$, $\phi(0) = \phi(\alpha) = 1$. Also for $0 < t < \alpha$,

$$\phi(t) = \phi(\lambda_1 0+\lambda_2\alpha) \leq \lambda_1 \phi(0)+\lambda_2\phi(\alpha) = 1, \text{ where } \lambda_1+\lambda_2 = 1,\ \lambda_1 > 0,\ \lambda_2 > 0.$$

Hence $\phi(t) = 1$ for $0 \leq t \leq \alpha$, and $\phi(t)$ is strictly increasing with t for $t \geq \alpha$.

By hypothesis, $x \perp y$ implies $||x+y|| = ||x-y||$ and $\alpha y+x \perp y$ implies $||(\alpha+1)y+x|| = ||(\alpha-1)y+x||$. Consequently, $\phi(1) = \phi(-1)$ and $\phi(\alpha+1) = \phi(\alpha-1)$, thus $0 < \alpha \leq 1$.

$$\begin{aligned}\text{Now, } \phi(\alpha-1) &= ||(\alpha-1)y+x|| = ||\tfrac{\alpha}{2}(x+y)+(1-\tfrac{\alpha}{2})(x-y)|| \\ &\leq ||x+y|| = \phi(1) = ||(1-\tfrac{\alpha}{2})((\alpha+1)y+x)+\tfrac{\alpha}{2}((\alpha-1)y+x)|| \\ &\leq \phi(\alpha-1).\end{aligned}$$

This implies that $\phi(\alpha-1) = \phi(1)$ - not possible. Hence X has to be strictly convex.

Lemma 4.2.3. If X satisfies (ii) then orthogonality is symmetric.

Proof. If not, let $x \perp y$ and $\alpha x+y \perp x$, $||x|| = ||y|| = 1$, $\alpha > 0$. If $\beta = ||\alpha x+y||$, then $1 = ||y|| = ||\alpha x+y-\alpha x|| \geq ||\alpha x+y|| = \beta \geq \alpha$. Let $||x+y|| = ||x-y|| = a$ and $||(\beta+\alpha)x+y|| = ||(\beta-\alpha)x-y|| = b$, then $b = ||(\beta-\alpha)x-y|| = ||\frac{\beta-\alpha-1}{2}(x+y) + \frac{\beta-\alpha+1}{2}(x-y)|| \leq a$ [Since $\beta = ||\alpha x+y|| \leq \alpha+1$ and $|\frac{\beta-\alpha-1}{2}| = \frac{1+\alpha-\beta}{2}$]. Similarly we can obtain $a \leq b$. Thus we have

$$||x+y|| = ||x-y|| = ||(\beta+\alpha)x+y|| = ||(\beta-\alpha)x-y||,$$

which is false since X is strictly convex.

Lemma 4.2.4. If X satisfies (ii) then it satisfies

(1) $$x,y \in S,\ x \perp y \implies ||x+y|| = ||x-y|| = \sqrt{2}.$$

Proof. Let $x,y \in S$ and $x \perp y$. Firstly, we will show that $x+y \perp x-y$. If not, let $x+ay \perp x-y$, where in view of the symmetry of orthogonality we may assume $0 < a < 1$. Let $||x+ay|| = \beta_1$ and $||x-y|| = \beta_2 = ||x+y||$, clearly $\beta_1 \leq \beta_2$.

We may further assume that X is a plane and introduce a coordinate system with $x = (1,0)$ and $y = (0,1)$. Then $x+ay = (1,a)$ and $x-y = (1,-1)$. Using the result of Day [5, p. 331] that $u = (u_1,u_2)$ and $v = (v_1,v_2)$ are orthogonal if and only if $|u_1v_2-u_2v_1| = ||u||\ ||v||$, we obtain $\beta_1\beta_2 = (1+a)$.

Moreover, from hypothesis we have

(2) $$||(\beta_2+\beta_1)x+(a\beta_2-\beta_1)y|| = ||(\beta_2-\beta_1)x+(a\beta_2+\beta_1)y||$$

which yields $\beta_2+\beta_1 \leq \beta_2-\beta_1+a\beta_2+\beta_1$, i.e. $\beta_1 \leq a\beta_2$

$$||x+y|| = \beta_2 = ||\frac{2}{1+a}(x+ay) + \frac{a-1}{a+1}(x-y)|| \leq \frac{2}{1+a}\beta_1 + \frac{1-a}{1+a}\beta_2$$

so that $a\beta_2 \leq \beta_1$.

In view of these inequalities, (2) becomes

$$\beta_2+\beta_1 = ||(\beta_2-\beta_1)x + 2\beta_1 y||$$

i.e.
$$1 = ||\frac{\beta_2-\beta_1}{\beta_2+\beta_1}x + \frac{2\beta_1}{\beta_2+\beta_1}y||$$

which contradicts strict convexity unless $\beta_1 = \beta_2$. But then $a = 1$. Thus $x+y \perp x-y$ and $||x+y|| = ||x-y|| = \sqrt{2}$.

Proof of Theorem 4.2.1.

Let $x \perp y$ be any pair of non-zero vectors.

Put
$$F(t) = \frac{||t^2y+x||}{t^2||y||+||x||}, \quad 0 \leq t < \infty.$$

$F(t)$ is differentiable because X is smooth-symmetry of orthogonality and strict convexity gives smoothness. Let $q'(x,y)$ denote the Gâteaux derivative of the norm at x in the direction of y. For an extreme value of F, we have

$$F'(t) = 0 \iff q'(t^2y+x,y) = \frac{||y||\ ||t^2y+x||}{t^2\ ||y||+||x||}$$

$$\iff \frac{||t^2y+x||}{t^2} - \frac{q'(t^2y+x,x)}{t^2} = \frac{||y||\ ||t^2y+x||}{t^2||y||+||x||}$$

$$\iff \frac{q'(t^2y+x,x)}{t^2} = ||t^2y+x||\left[\frac{t^2||y||+||x||-t^2||y||}{t^2(t^2||y||+||x||)}\right]$$

$$\iff q'(t^2y+x,x) = \frac{||x||\ ||t^2y+x||}{t^2||y||+||x||}\ .$$

This gives $q'(t^2y+x, ||y||x-||x||y) = 0$, i.e. $t^2y+x \perp ||x||y-||y||x$.

This shows that there is only one extreme value of $F(t)$ for $t > 0$, which by Lemma 4.2.3 corresponds to $t^2 = \frac{||x||}{||y||}$ and the extreme value is

$$F(t)\Big|_{extreme} = \frac{||\ ||x||y+||y||x||}{2||x||\ ||y||} = \frac{1}{\sqrt{2}}\ ,$$

which must be minimum. Thus $\frac{||x||+||y||}{||x+y||} \leq \sqrt{2}$ whenever $x \perp y$.

Then the rectangular constant

$$m(X) = \sup_{x \perp y} \frac{||x||+||y||}{||x+y||} \leq \sqrt{2}.$$

Hence $m(X) = \sqrt{2}$, which was to be proved.

We now give refinements of some of the earlier known results of Day [6], Kapoor and Prasad [22] and Holub [18] as corollaries to our main theorem.

Corollary 4.2.5. A normed linear space X is an inner product space if it has any one of the following properties :

(i) $x,y \in S$, $x \perp_i y \implies x \perp y$

(ii) $x,y \in S$, $x \perp y \implies x \perp_p y$

(iii) $x,y \in S$, $x \perp_p y \implies x \perp y$.

Proof (i) Assuming the space is not strictly convex, choose $x,y \in S$ and the largest real $\eta > 0$ such that $\beta y+x \perp y$, $\beta y+x \in S$, for all $-\eta \leq \beta \leq \eta$. We claim that $x \perp_i y$.

The function $||x+z|| - ||x-z||$ varies continuously between -2 and 2, as z moves from $-x$ to x along the curve S_1, which is the intersection of S and the span of x and y. Hence there is a $z = ax+by$, $b > 0$, in S_1, such that $||x+z|| = ||x-z||$.

The hypothesis implies that $x \perp ax+by$, and

$$1 = ||x \pm \eta y|| = ||\frac{b \mp a\eta}{b} x \pm \frac{\eta}{b}(ax+by)|| \geq |\frac{b \mp a\eta}{b}|.$$

Thus $b \geq |b \pm a\eta|$. This leads to a contradiction unless $a = 0$ and $b = 1$ and thus $x \perp_i y$.

Similarly it can be proved that $\beta y+x \perp_i y$ for $-\eta \leq \beta \leq \eta$. It is easily seen that $\eta \leq 1$.

Putting $\phi(t) = ||ty+x||$ we note that $\phi(t)$ is a convex function with $\phi(t) \geq 1$, $\phi(-\eta) = \phi(\eta) = 1$, $\phi(1) = \phi(-1)$. $\phi(t)$ is strictly decreasing for $-\infty < t \leq -\eta$ and strictly increasing for $\eta \leq t < \infty$, and $\phi(\eta+1) = \phi(\eta-1)$, which is not possible since $\eta+1 > 1$ and $-1 \leq \eta-1 \leq 0$. Therefore, the space must be strictly convex.

To complete the proof we show that our hypothesis implies the hypothesis of Theorem 4.2.1. Let $x,y \in S$ and $x \perp y$, choose a and b as above such that $ax+by \perp_i y$, with $a \geq 0$, hence $ax+by \perp y$ and from strict convexity it follows that $a = 1$ and $b = 0$.

The proof of (ii) is immediate from Theorem 4.2.1, and (iii) ==> (ii) can be proved following the lines of proof of (i) above.

Corollary 4.2.6. If $x, y \in S$, $x \perp y \Longrightarrow ||x+y||^2+||x-y||^2 = 4$ then X is an inner product space, provided orthogonality is assumed to be symmetric.

Proof. Let $x, y \in S$ and $x \perp y$. Let $x+ay \perp x-ay$ [29], where $0 < a \leq 1$. Put $||x+ay|| = \beta_1$ and $||x-ay|| = \beta_2$. The hypothesis gives

$$||(\beta_2+\beta_1)x+(\beta_2-\beta_1)ay||^2+||(\beta_2-\beta_1)x+(\beta_2+\beta_1)ay||^2 = 4\beta_1^2\beta_2^2$$

and $||x+y||^2 + ||x-y||^2 = 4$, whence

(1) $$4\beta_1^2\beta_2^2 \geq (\beta_2+\beta_1)^2 \ (1+a^2).$$

Also $||x+y|| = ||\frac{a+1}{2a}(x+ay) + \frac{a-1}{2a}(x-ay)||$ gives $||x+y|| \geq \frac{a+1}{2a}\beta_1$.

Similarly $||x-y|| \geq \frac{a+1}{2a}\beta_2$ and, therefore

(2) $$\left(\frac{a+1}{2a}\right)^2 (\beta_1^2+\beta_2^2) \leq 4$$

(1) and (2) $\Longrightarrow a = 1$ and $\beta_1 = \beta_2 = \sqrt{2}$.

By the main theorem we get the result.

4.3 A sufficient condition for strict convexity

If the symmetry of orthogonality is not there in Corollary 4.2.6 we do not know how to prove it, though we feel it should be true. Without symmetry we can still prove that the space is strictly convex in the following

Theorem 4.3.1. If $x \perp y \Longrightarrow ||x+y||^2+||x-y||^2 = 2[||x||^2+||y||^2]$ then X is strictly convex.

<u>Proof</u>. Let $||x|| = ||y|| = ||\frac{x+y}{2}|| = 1$. It can be easily seen that $\frac{x+y}{2} \not\perp x$. Choose α such that $\frac{x+y}{2} \perp \alpha \frac{x+y}{2} + x$, but then

$$1 = ||\frac{x+y}{2}|| \leq ||\frac{x+y}{2} + \lambda\alpha \frac{x+y}{2} + \lambda x||, \quad \lambda \in R.$$

Putting $\lambda = -\frac{1}{\alpha}$ and $\lambda = -\frac{1}{\alpha+2}$ we get $|\alpha| \leq 1$ and $|\alpha+2| \leq 1$, which gives $\alpha = -1$, hence $x+y \perp x-y$. Now we must have

$$\begin{aligned} 8 &= ||x+y+x-y||^2 + ||x+y-x+y||^2 \\ &= 2\left[||x+y||^2 + ||x-y||^2\right] \\ &= 8 + 2||x-y||^2, \end{aligned}$$

which implies that $x = y$, and the proof is complete.

CHAPTER V

METRIC PROJECTION BOUND AND THE LIPSCHITZ CONSTANT OF THE RADIAL RETRACTION

5.1 Introduction. Looking at Theorems 5.1.1 and 5.1.3 about symmetry of orthogonality and Theorems 5.1.2 and 5.1.4 characterizing uniformly non-square Banach spaces, we were led to the equality of the two constants MPB(X) and k(X) of a normed linear space X. Our main result in this chapter is the proof of this equality. In the end we have given an expression for the metric projection bounded of the space ℓ_p^2, and then we have calculated the numerical values of the same for ℓ_3^2 and ℓ_4^2.

We recall the definitions given in 0.2.16. Let X be a real normed linear space, M its non-trivial closed proper subspace. The (possibly empty) set of best approximations to x from M is defined by

$$P_M(x) = \{y \in M : ||x-y|| = d(x,M)\}$$

where $d(x,M) = \inf \{||x-m|| : m \in M\}$. The subspace M is called proximinal if $P_M(x)$ contains atleast one point for every $x \in X$. The mapping $P_M : X \to 2^M$ is called the metric projection on M. If M is proximinal, the norm of P_M is defined by

$$||P_M|| = \text{Sup } \{||y|| : y \in P_M(x), ||x|| \leq 1\}$$

It is easily seen that $1 \leq ||P_M|| \leq 2$ for every proximinal subspace M of X. The metric projection bound of X written as

MPB(X) is defined to be

$$MPB(X) = \text{Sup}\ \{||P_M|| : M \text{ proximinal subspace of } X\}.$$

If X is a Hilbert space then MPB(X) = 1. In general, $1 \leq MPB(X) \leq 2$. Deutsch and Lambert [12] have constructed a Chebysev subspace in C[0,1] whose metric projection is linear and has norm two. Smith [28] has proved the following two theorems :

Theorem 5.1.1. The orthogonality is symmetric in a normed linear space X if and only if MPB(X) = 1.

Theorem 5.1.2. A Banach space X is uniformly non-square if and only if MPB(X) < 2.

We also know

Theorem 5.1.3. (de Figueiredo and Karlovitz [14]). The orthogonality is symmetric in a normed linear space X if and only if the Lipschitz constant k(X) of radial retraction is one.

Theorem 5.1.4 (Thele [31]). A Banach space X is uniformly non-square if and only if k(X) < 2.

Theorem 5.1.5 (Thele [31])

$$k(X) = \text{Sup}\ \{\frac{||y||}{||y-\lambda x||} : y \neq 0,\ x \perp y,\ \lambda \varepsilon R\}.$$

5.2 Metric projection bound and symmetry of orthogonality

If X is a Hilbert space then orthogonality is symmetric and MPB(X) = 1. In general we have the Theorem 5.1.1 which has recently been proved by Smith [28], but we give its proof for completion sake.

Proof of Theorem 5.1.1

If the orthogonality is symmetric and dimension $X \geq 3$, then X is an inner product space and therefore $MPB(X) = 1$. Suppose dimension $X = 2$. Let M be a proper nontrivial subspace of X. M is the span $[y]$ of some unit vector y. Let $x \in X$ and P_y denote the metric projection on M. We then have

$$\begin{aligned} \alpha y \in P_y(x) &\iff x-\alpha y \perp y \\ &\iff y \perp x-\alpha y \\ &\iff \text{there is } f \in S(X^*) \text{ such that} \\ &\qquad f(y) = ||y|| = 1 \text{ and } f(x-\alpha y) = 0. \end{aligned}$$

From this it follows that

$$P_y(x) = \{f(x)y : f \in S(X^*),\ f(y) = 1\}.$$

Therefore, $||P_y|| = 1$ and $MPB(X) = 1$.

Suppose on the other hand that $MPB(X) = 1$. Let x and y any two non-zero vectors. It is easily verified that $x-y \perp y$ implies that $y \in P_y(x)$. Now assume $x \perp y$, but then $tx+y-y \perp y$ for any real t. Therefore $y \in P_y(tx+y)$. Now $MPB(X) = 1$ implies $||y|| \leq ||tx+y||$ for any real t. Hence $y \perp x$.

5.3 Metric projection bound and Lipschitz constant of radial retraction

Theorem 5.3.1. For any normed linear space X, the metric projection bound MPB(X) and the Lipschitz constant k(X) are equal.

To prove the theorem we first prove the following

Lemma 5.3.2. $MPB(X) = \text{Sup}\,\{||P_y|| : y \in X\}$.

Proof. Clearly $\text{Sup}\,\{||P_y|| : y \in X\} \leq MBP(X) = m(\text{say})$. Let $\varepsilon > 0$, choose M a proximinal subspace such that $||P_m|| > m-\varepsilon$, then there exists $x \in X$ and $y \in P_M(x)$ such that $||y|| > m-\varepsilon$. Also $||x-y|| \leq ||x-z||$ for every $z \in M$, therefore, $||x-y|| \leq ||x-ty||$ for every $t \in R$, and hence $y \in P_y(x)$. Then $||P_y|| \geq ||y|| > m-\varepsilon$, which proves the lemma.

Proof of Theorem 5.3.1.

Let $x \perp y$. Then $y \in P_y(x+y)$ and therefore,

$$\frac{||y||}{||x+y||} \leq ||P_y|| \leq MPB(X),$$

which implies that

$$\text{Sup}\,\{\frac{||y||}{||x+y||} : x \perp y\} \leq MPB(X) = m \text{ (say)}.$$

On the other hand if $\varepsilon > 0$, choose $y \in X$ such that $||P_y|| > m-\varepsilon$. Let z be such that $m-\varepsilon \leq \frac{||b||}{||z||}$ for some $b \in P_y(z)$ then $b = ty$ for some $t \in R$ and hence $b \in P_b(z)$, giving that $z-b \perp b$. Thus $m-\varepsilon \leq \text{Sup}_{x \perp y} \frac{||y||}{||x+y||}$, hence $m = \text{Sup}_{x \perp y} \frac{||y||}{||x+y||}$.

Using the result of Theorem 5.1.5 that

$$k(X) = \text{Sup}\,\{\frac{||y||}{||\alpha x-y||} : y \neq 0,\ x \perp y,\ \alpha \in R\}$$

$$= \text{Sup}\,\{\frac{||y||}{||x+y||} : x \perp y\},$$

we get the result of the theorem.

5.4 Metric projection bound for ℓ_p^n.

It is easily seen that $MPB(\ell_p^n) = 2$ if $p = 1$ or ∞. If $p \neq 1$ or ∞ then ℓ_p is smooth. The normalized duality map J

is given by $J(\theta) = 0$ and $J_x = \left(\frac{|x_i|^{p-1} \operatorname{sgn} x_i}{||x||^{p-2}}\right)$, for $\theta \neq x=(x_i)$.

If $\theta \neq x$ and $y = (y_i)$ then $x \perp y$ if and only if

$$(J_x, y) = \sum_{i=1}^{n} \frac{|x_i|^{p-1} y_i \operatorname{sgn} x_i}{||x||^{p-2}} = 0.$$

By Theorem 5.3.1, $MPB(X) = k(X) = \sup\limits_{x \perp y} \frac{||y||}{||x+y||}$.

It is easily seen that

$$\frac{1}{k(X)} = \inf_{\substack{\alpha \in R \\ x \perp y \\ ||y||=1}} ||\alpha x - y|| = \inf_{\substack{||y||=1 \\ x \perp y \\ \alpha x - y \perp x}} ||\alpha x - y||$$

This means that for ℓ_p^n we have to find the minimum value of $||\alpha x - y||_p$ under the constraints that

$$\sum_{i=1}^{n} |y_i|^p = 1$$

$$(5.4.1) \qquad \sum_{i=1}^{n} y_i |x_i|^{p-1} \operatorname{sgn} x_i = 0$$

$$\sum_{i=1}^{n} x_i |\alpha x_i - y_i|^{p-1} \operatorname{sgn} (\alpha x_i - y_i) = 0.$$

These conditions become unwieldy in the general case, but give a nice result in the two-dimensional case. We give

the result in the following theorem, wherein we will use the notation $||x||_r = (\Sigma\, |x_i|^r)^{1/r}$ even when $0 < r < 1$.

<u>Theorem 5.4.2.</u> For $1 < p < \infty$

$$MPB(\ell_p^2) = k(\ell_p^2) = \sup_{x \in \ell_p^2} \left[\frac{||x||_{p(p-1)}^{p-1}\, ||x||_q}{||x||_p^p} \right].$$

<u>Proof</u>. The constraints (5.4.1) for the two-dimensional case are

(i) $|y_1|^p + |y_2|^p = 1$

(ii) $y_1 |x_1|^{p-1} \operatorname{sgn} x_1 + y_2 |x_2|^{p-1} \operatorname{sgn} x_2 = 0$

(iii) $x_1 |\alpha x_1 - y_1|^{p-1} \operatorname{sgn}(\alpha x_1 - y_1) + x_2 |\alpha x_2 - y_2|^{p-1} \operatorname{sgn}(\alpha x_2 - y_2) = 0.$

We assume that $0 < x_1 < 1$, $0 < x_2 < 1$, $y_1 < 0$, $y_2 > 0$ and $\alpha > 0$, and the other cases are similarly dealt with.

$$\begin{aligned} ||\alpha x - y||_p^p &= ||\alpha x - y||^{p-2} (J_{\alpha x - y}, \alpha x - y) \\ &= ||\alpha x - y||^{p-2} (J_{\alpha x - y}, -y) \\ &= -(y_1 |\alpha x_1 - y_1|^{p-1} \operatorname{sgn}(\alpha x_1 - y_1) \\ &\qquad + y_2 |\alpha x_2 - y_2|^{p-1} \operatorname{sgn}(\alpha x_2 - y_2)). \end{aligned}$$

Putting the value of $|\alpha x_2 - y_2|^{p-1} \operatorname{sgn}(\alpha x_2 - y_2)$ from (iii)

$$||\alpha x - y||_p^p = \frac{y_2 x_1 - y_1 x_2}{x_2} |\alpha x_1 - y_1|^{p-1} \operatorname{sgn}(\alpha x_1 - y_1)$$

and $x_1(\alpha x_1 - y_1)^{p-1} = x_2(y_2 - \alpha x_2)^{p-1}$, which yields

$$\alpha = \frac{y_1 \; x_1^{\frac{1}{p-1}} + y_2 \; x_2^{\frac{1}{p-1}}}{x_1^{\frac{p}{p-1}} + x_2^{\frac{p}{p-1}}} \; .$$

We can re-write condition (ii) as $-y_1 x_1^{p-1} = y_2 x_2^{p-1}$ and combining this with (i) we finally get

$$x_1 y_2 - x_2 y_1 = \frac{x_1^p + x_2^p}{(x_1^{p(p-1)} + x_2^{p(p-1)})^{\frac{1}{p}}} \, ,$$

and $$||\alpha x - y||_p = \frac{|x_1|^p + |x_2|^p}{(|x_1|^q + |x_2|^q)^{\frac{1}{q}} \, (|x_1|^{p(p-1)} + |x_2|^{p(p-1)})^{\frac{1}{p}}}$$

$$= \frac{||x||_p^p}{||x||_q \; ||x||_{p(p-1)}^{p-1}} \; .$$

From this the result follows.

Remark. The above theorem raises the following questions about norm inequalities in ℓ_p spaces

4.3) Is $k(\ell_p^n) = \sup\limits_{x \, \varepsilon \, \ell_p^n} \dfrac{||x||_{p(p-1)}^{p-1} \; ||x||_q}{||x||_p^p}$?

the answer is yes then we shall have

$$1 \leq \frac{||x||_{p(p-1)}^{p-1} \; ||x||_q}{||x||_p^p} \leq 2$$

.4) Is $1 \leq \dfrac{||x||_{p(p-1)}^{p-1} \; ||x||_q}{||x||_p^p} \leq 2$ for $x \, \varepsilon \, \ell_p$ or ℓ_p^n ?

The first inequality in (5.4.4) is of-course true for any ℓ_p, as it follows from the convexity of the function $f(r) = \log ||x||_r^r$ for $0 < r < \infty$. Putting $p = \frac{1}{p}\, p(p-1) + (1 - \frac{1}{p})q$,

$$\log ||x||_p^p \leq \frac{1}{p} \log ||x||_{p(p-1)}^{p(p-1)} + (1 - \frac{1}{p}) \log ||x||_q^q$$

and so $||x||_p^p \leq ||x||_{p(p-1)}^{p-1} \; ||x||_q$.

We can-not say for definite that the second inequality in (5.4.4) holds for all ℓ_p, but for ℓ_p^n and when p is sufficiently close to 2, it can be proved as follows:

<u>Case I.</u> $2 < p < \infty$, then $q < p < p(p-1)$, hence

$$||x||_{p(p-1)} \leq ||x||_p \text{ and } 1 \leq \frac{||x||_q}{||x||_{p(p-1)}} \leq n^{\frac{1}{q} - \frac{1}{p(p-1)}} .$$

From the second inequality,

$$||x||_{p(p-1)}^{p-1} \; ||x||_q \leq n^{\frac{p-2}{p-1}} \; ||x||_{p(p-1)}^p$$

$$\leq 2||x||_p^p \text{ , when p is sufficiently close to 2.}$$

<u>Case II.</u> $1 < p < 2$, then $p(p-1) < p < q$, hence as above

$$||x||_{p(p-1)} \leq n^{\frac{2-p}{p(p-1)}} \; ||x||_p, \text{ and from this}$$

$$||x||_{p(p-1)}^{p-1} \; ||x||_q \leq n^{\frac{2-p}{p}} \; ||x||_p^{p-1} \; ||x||_q$$

$$\leq 2||x||_p^p, \text{ when p is very nearly 2.}$$

We now proceed to find an expression for $k(\ell_p^2)$ and from which we calculate the values of $k(\ell_3^2)$ and $k(\ell_4^2)$. For p = 5,6 etc. the equations obtained can-not be solved by algebraic methods.

Take $x = (x_1, x_2) \in \ell_p^2$ and put $t = \frac{|x_2|}{|x_1|}$ assuming $|x_1| \geq |x_2|$. Let

$$A = \frac{||x||_{p(p-1)}^{p-1} \; ||x||_q}{||x||_p^p} = \frac{(|x_1|^{p(p-1)}+|x_2|^{p(p-1)})^{\frac{1}{p}}(|x_1|^q+|x_2|^q)^{\frac{1}{q}}}{|x_1|^p + |x_2|^p}$$

$$= \frac{(1+t^{p(p-1)})^{\frac{1}{p}} \; (1+t^q)^{\frac{1}{q}}}{1+t^p} .$$

For a maximum value of A, $\frac{dA}{dt} = 0$, hence t is given by

$$p(1+t^q)(t^{p(p-1)}-t^p) - (1+t^p)(t^{p(p-1)}-t^q) = 0.$$

Dividing by t^q, and putting $t^q = z$,

$$p(1+z) \; (z^{p-2}-z^{p(p-2)}) = (1+z^{p-1})(1-z^{p(p-2)}).$$

<u>When p = 3</u>

$$3(1+z) \; (z-z^3) = (1+z^2)(1-z^3).$$

This reduces to a symmetric equation in z, putting $z + \frac{1}{z} = y$, we get $y^2-2y-6 = 0$. Its admissible solution is $y = 1 + \sqrt{7}$, hence

$$A_{max} = \frac{(1+t^6)^{\frac{1}{3}} \; (1+t^{\frac{3}{2}})^{\frac{2}{3}}}{1+t^3}$$

$$= \frac{(1+z^4)^{\frac{1}{3}} \; (1+z)^{\frac{2}{3}}}{1+z^2}$$

$$= \frac{\left(z^2 + \frac{1}{z^2}\right)^{\frac{1}{3}} \left(z + 2 + \frac{1}{z}\right)^{\frac{1}{3}}}{\left(z + \frac{1}{z}\right)}$$

$$= \frac{(17 + 7\sqrt{7})^{\frac{1}{3}}}{3} .$$

When $p = 4$

$$4(1+z)\ z^2\ (1-z^6) = (1+z^3)(1-z^8) .$$

This again reduces to symmetric form, putting $z + \frac{1}{z} = y$, we get the equation $y^3-6y-4 = 0$ which is $(y+2)(y^2-2y-2) = 0$ giving $y = 1 + \sqrt{3}$. Substituting the values

$$A_{max} = \frac{(1+t^{12})^{\frac{1}{4}}\ (1+t^{\frac{4}{3}})^{\frac{3}{4}}}{1+t^4}$$

$$= \frac{(1+z^9)^{\frac{1}{4}}\ (1+z)^{\frac{3}{4}}}{1+z^3}$$

$$= \frac{(1-z^3+z^6)^{\frac{1}{4}}}{(1-z+z^2)^{\frac{3}{4}}}$$

$$= (1 + \tfrac{2}{3}\sqrt{3})^{\frac{1}{4}} .$$

When $p = 5$ and $p = 6$, we get equations of eighth and tenth degree respectively, which can-not be solved by algebraic methods.

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